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TOPOLOGICAL REPRESENTATION OF LUKASIEWICZ AND POST ALGEBRAS

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Roberto Cignoli

The aim of this paper is to give a construction of the dual category of n-valued Lukasiewicz algebras. This construction is based on a representation of Lukasiewicz algebras by certain Lukasiewicz algebras of subsets of the set of prime filters (section 1). The main results are obtained endowing the set of prime filters with the Stone topology (section 2). The particular case of Post algebras is briefly considered in section 3.

The results of sections 1 and 2 were contained in the author's doctoral thesis, written under the advice of Professor Antonio Monteiro and submitted to the Universidad Nacional del Sur in October, 1969. They have been used by M. Fidel [7] to give a semantic interpretation of n-valued Lukasiewicz algebras, and by I.Petrescu ([12], see remark following Corollary 2.5) to give a construction of the dual category of the category of DeMorgan algebras; and they have been generalized by G.Georgescu [8] to infinite valued Lukasiewicz algebras.

n-valued Lukasiewicz algebras were introduced by G. Moisil in 1941 [9]. The Moisil's papers on the subject are collected in [10], and the recent work done by Moisil's students can be found in [3]. This paper has to be considered as a continuation of the author's papers [4] and [5], and so, the definitions and notations are as in these papers. For completeness, we are going to reproduce here the definition of n-valued Lukasiewicz algebras (named Moisil algebras in [4], see the footnote of [5]).

A DeMorgan algebra (Quaşi-Boolean algebra in the nomenclature of [1]) is a system $(A,1,\vee,\wedge,-)$ such that $(A,1,\vee,\wedge)$ is a distributive lattice with unit 1 and - is a unary operation defined on A fulfilling the conditions:

M1) $--x = x$ and M2) $-(x \vee y) = -x \wedge -y$

Note that $-1 = 0$ is the zero of the lattice $(A,\vee,\wedge)$.

An n-valued Lukasiewicz algebra (n an integer $\geqslant 2$) is a system $(A,1,\vee,\wedge,-,s_1,\ldots,s_{n-1})$ such that $(A,1,\vee,\wedge,-)$ is a DeMorgan algebra and s_i $(1 \leqslant i \leqslant n-1)$ are unary operations defined on A fulfilling the conditions:

L1) $s_i(x \vee y) = s_i x \vee s_i y$

L2) $s_i x \vee -s_i x = 1$

L3) $s_i s_j x = s_j x$

L4) $s_i -x = -s_{n-i} x$

L5) $s_1 x \leqslant s_2 x \leqslant \ldots \leqslant s_{n-1} x$

L6) If $s_i x = s_i y$ for $i = 1,\ldots,n-1$, then $x = y$.

1. REPRESENTATION BY SETS.

In what follows, A will denote a fixed n-valued Lukasiewicz algebra.

Let $E = E(A)$ be the set of all prime filters of A, considered as a lattice. Since A is a DeMorgan algebra, we can define, following Bialynicki-Birula and Rasiowa [1], a function $g_o \colon E \longrightarrow E$ by the formula

(1) $$g_o(P) = C - P$$

where P is in E, C denotes the set theoretical complement, and $-P = \{-x: x \in P\}$.

On the other hand, since the operations s_i $(i=1,\dots,n-1)$ are lattice homomorphisms from A into A, it follows that we can define n-1 functions $g_i: E \longrightarrow E$ $(i=1,\dots,n-1)$ by the formulae:

(2) $g_i(P) = s_i^{-1}(P) = \{x \in A: s_i x \in P\}, \; 1 \leqslant i \leqslant n-1.$

1.1. LEMMA. *Let* $E = E(A)$ *and let* $g_i: E \longrightarrow E$ *be the functions defined by* (1) *for* $i=0$ *and by* (2) *for* $i=1,2,\dots,n-1$. *Then the following properties hold:*

A1) $g_o g_o = I =$ *identity on* E.

A2) $g_i g_j = g_i$ *for* $i=1,\dots,n-1$ *and* $j=0,1,\dots,n-1$

A3) $g_o g_i = g_{n-i}$ *for* $i=1,\dots,n-1$

A4) $E = g_1(E) \cup \dots \cup g_{n-1}(E)$

PROOF. A1) is a well known result in the theory of DeMorgan algebras [1], A2) and A3) follow at once from the definitions of the functions g_i. To prove A4), let P be a prime filter of A. Then $P^* = P \cap B(A)$ is a prime filter of B(A) and there exists an i $(1 \leqslant i \leqslant n-1)$ such that $P = P_i^* = \{x \in A: s_i x \in P^*\}$ ([4], Th. 4.7). On the other hand, $x \in g_i(P)$ is equivalent to $s_i x \in P$, and since $s_i x \in B(A)$, it is equivalent to $s_i x \in P^*$, therefore, $g_i(P) = P_i^* = P$. So $P \in g_i(E)$, and A4) is proved.

REMARKS. 1) The map $P^* \longmapsto P^*$ is a bijection from the set M of all ultrafilters of B(A) into $E_i = g_i(E)$. For, as we shown in the proof of A4), $g_i(P_i^*) = P_i^*$, so $P^* \longmapsto P_i^*$ is a map from M into E_i. It is one-to-one, because if $P_i^* = Q_i^*$, then $P^* = P_i^* \cap B(A) = Q_i^* \cap B(A) = Q^*$; and it is onto because if $Q \in E_i$, then $Q = g_i(P)$ for certain P in E. Setting $P^* = P \cap B(A)$, we have $g_i(P) = P_i^*$, hence, $Q = P_i^*$.

2) If $E_i = g_i(E)$ $(i=1,\ldots,n-1)$, then $E_i \cap E_j \neq \emptyset$ if and only if there exists P^* in M such that $P_i^* = P_j^*$. For, $P \in E_i \cap E_j$ if and only if there exist P^*, Q^* in M such that $P = P_i^* = Q_j^*$. But this implies $P^* = Q^*$, therefore, $P_i^* = P_j^*$. Conversely, if $P_i^* = P_j^* = P$, we have $g_i(P) = g_j(P) = P \in E_i \cap E_j$.

From 2) and [4], p 26, it follows at once that:

3) $E_i \cap E_j = \emptyset$ for $i \neq j$ $(1 \leqslant i,j \leqslant n-1)$ if and only if all the maximal deductive systems of A are of order n.

4) A4) and preceding remarks allow us to establish that the set E is the set theoretical union of n-1 sets equipotent to M, but as they are not necessarily disjoint, in general E is smaller than the sum of the sets E_i. E is the sum of the sets E_i if and only if all the maximal deductive systems of A are of order n (cf. [4], Prob. 8.9).

Assume now that E is a set and $g_o, g_1, \ldots, g_{n-1}$ are n functions from E into E that fulfill the conditions A1) - A4) of Lemma 1.1. If $X \subseteq E$, define:

(3) $\quad -X = C\, g_o(X)$

(4) $\quad s_i(X) = g_i^{-1}(X)$, $i = 1,\ldots,n-1$.

It is well known [1] that the system $(2^E, E, \cup, \cap, -)$ is a DeMorgan algebra, and we have also:

1.2. LEMMA. *The operations* $-$ *and* s_i $(i=1,\ldots,n-1)$ *defined by* (3) *and* (4) *respectively fulfill the properties* L1), L2), L3), L4) *and* L6) *of the definition of Lukasiewicz algebras.*

PROOF. L1) follows at once from the properties of the inverse image. From A1) and A2) it follows that $g_i^{-1}(X) = g_o(g_i^{-1}(X))$, and taking set theoretical complements, we get:

$$\text{(i)} \qquad C\, s_i(X) = -s_i(X)$$

which implies L2).

L3) follows at once from A2). In order to prove L4), we begin observing that A1), A3) and the properties of the inverse image imply:

$$\text{(ii)} \quad s_i(-X) = g_i^{-1}(g_o(C\, X)) = (g_o g_i)^{-1}(C\, X) = s_{n-i}(C\, X)$$

On the other hand, using the properties of the inverse image, we obtain:

$$\text{(iii)} \qquad s_j(C\, X) = C\, s_j(X) \quad , \quad j=1,\ldots,n-1.$$

L4) follows from (ii), (iii) and (i).

In order to prove L6), suppose that $s_i(X) = g_i^{-1}(X) = g_i^{-1}(Y) = s_i(Y)$ for $i=1,\ldots,n-1$, and let $x \in X$. A4) implies that there is an z in E and an i $(1 \leqslant i \leqslant n-1)$ such that $x = g_i(z)$. Therefore, $z \in g_i^{-1}(X) = g_i^{-1}(Y)$, and there is an y in Y such that $y = g_i(z)$. Hence, $x = y \in Y$, and we have proved that $X \subseteq Y$. Interchanging X and Y we obtain L6), and the proof of lemma is completed.

1.3. LEMMA. *The system* $(2^E, E, \cup, \cap, -, s_1, \ldots, s_{n-1})$, *where the operations* $-$ *and* s_i *are defined by formulae* (3) *and* (4) *respectively, is a n-valued Lukasiewicz algebra if and only if* $g_o = g_1 = \ldots = g_{n-1} = I$ = *identity on* E; *and in this case it reduces to the Boolean algebra of all subsets of the set* E.

PROOF. From the above lemma, it follows that we have to prove that property L5) is fulfilled if and only if it is fulfilled the condition:

(i) $g_o = g_1 = \ldots = g_{n-1} = I$

It is clear that (i) implies that $-X = C\,X$, $s_i X = X$ $(i=1,\ldots,n-1)$, so $(2^E, E, \cup, \cap, -, s_1, \ldots, s_{n-1})$ is the Boolean algebra of all subsets of E with the natural operations, and a fortiori it is a n-valued Lukasiewicz algebra for $n \geqslant 2$.

Conversely, suppose that L5) holds, and let $z \in E$. We set $g_1(z) = x$ and $X = \{x\}$. Then we have $s_1(X) = g_1^{-1}(X) \subseteq$ $\subseteq g_i^{-1}(X) = s_i(X)$ for $i=2,3,\ldots,n-1$, and since z is in $g_1^{-1}(X)$, it follows that z is also in $g_i^{-1}(X)$, i.e., $g_i(z) \in X = \{x\}$. Hence $g_1(z) = x = g_i(z)$, and we have proved that $g_1 = g_2 = \ldots = g_{n-1}$. This equality, together with A4) imply that $E = g_i(E)$ for $i=1,\ldots,n-1$. So, any x in E is of the form $x = g_i(z)$ for some z in E, and using A3) we get $g_o(x) = g_o(g_i(z)) = g_{n-i}(z) = x$, i.e., $g_o = I$. Therefore, $-X = C\,X$, and the system $(2^E, E, \cup, \cap, -)$ is a Boolean algebra. From the Theorem 1.9 of [4] it follows at once that $s_i(X) = X$ for all $X \subseteq E$, i.e., that $g_i^{-1}(X) = X$. In particular, taking $X = \{x\}$, we get that $g_i(z) = x$ if and only if $x=z$. So we also have $g_1 = \dots = g_{n-1} = I$, and the proof is completed.

We are going to prove that, despite the above lemma, certain subalgebras of the system $(2^E, E, \cup, \cap, -, s_1, \dots, s_{n-1})$ are the most general examples of n-valued Lukasiewicz algebras.

1.4. DEFINITION. *Let* E *be a non-empty set, and* g_o, $g_1, \dots, g_{n-1}$ n *functions from* E *into* E *fulfilling the conditions* A1) - A4) *of Lemma* 1.1. *For any* $X \subseteq E$, *define* $-X$ *and* s_iX $(i=1,\dots,n-1)$ *by the formulae* (3) *and* (4) *respectively. Any subalgebra of the algebra* $(2^E, E, \cup, \cap, -, s_1, \dots, s_{n-1})$ *that satisfies the axiom* L5) *will be called a n-valued Lukasiewicz algebra of sets.*

We can now establish the main result of this section:

1.5. THEOREM. *Any n-valued Lukasiewicz algebra is isomorphic to a n-valued Lukasiewicz algebra of sets.*

PROOF. Let E be the set of all prime filters of the lattice A. Define g_o and g_i $(i=1,\dots,n-1)$ by the formulae (1) and (2) respectively. From Lemma 1.1, we know that $g_o, g_1, \dots, g_{n-1}$ fulfill the properties A1)-A4),and so if $-X$ and s_iX are defined by formulae (3) and (4) respectively,from Lemma 3 we know that the system $(2^E, E, \cup, \cap, -, s_1, \dots, s_{n-1})$ satisfies all the axioms of a n-valued Lukasiewicz algebra,

but L5).
On the other hand, we can define the Stone transformation $S: A \longrightarrow 2^E$ by the formula:

$$S(x) = \{P \in E: x \in P\}$$

and it is well known [1], that S is an homomorphism from the DeMorgan algebra A into the DeMorgan algebra $(2^E, E, \cup, \cap, -)$. Furthermore, from the definitions of the g_j and s_i it follows at once that $s_i S(x) = S(s_i x)$ for $i=1,\ldots,n-1$. Therefore S is a homomorphism from the n-valued Lukasiewicz algebra A into the similar algegra $(2^E, E, \cup, \cap, -, s_1, \ldots, s_{n-1})$ and since the n-valued Lukasiewicz algebras form an equational class (see [4], Remark 1.5) it follows from well known results of universal algebra (see, for instance, [2] Chapter VI) that the image $S(A) = A'$ is a subalgebra of $(2^E, E, \cup, \cap, -, s_1, \ldots, s_{n-1})$ which is a n-valued Lukasiewicz algebra.

Since S is one-to-one, we have proved that A is isomorphic to the n-valued Lukasiewicz algebra of sets A'.

2. TOPOLOGICAL REPRESENTATION.

In this section we are going to complete the above results endowing the set of prime filters of A with the Stone topology. In this form we will be able to give a description of the dual category of the category of n-valued Lukasiewicz algebras and homomorphisms.

First of all, we are going to recall some well known results of the theory of representation of distributive lattices, with the aim of further reference. We follow the exposition of A.Nerode [11].

2.1. DEFINITIONS. *A compact* T_0 *topological space is called a Stone space if the following conditions hold:*

S1) *The compact open sets form a multiplicative basis for the open sets.*

S2) *If* C *is a family of compact open sets with the finite intersection property and if* C *is a closed set such that* $F \cap C \neq \emptyset$ *for any* C *in* C*, then*
$$F \cap \bigcap_{C \in C} C \neq \emptyset.$$

If X *and* Y *are Stone spaces, a function* $f: X \longrightarrow Y$ *is called strongly continuous if the inverse image of a compact open set of* Y *is a compact open set of* X.

Observe that since the compact open sets form an open basis, any strongly continuous function in continuous.

If X is a set and $L \subseteq X$, we denote by U(L) the set of all subsets of X containing L. By the weak topology on 2^X is meant the topology with open basis consisting of U(L) for finite L.

2.2. THEOREM. (Stone [13]). *Let* L *be a distributive lattice with* 0 *and* 1*, and* L* *be the set of all prime filters of* L *considered as a weak subspace of* 2^L. *Then* L* *is a Stone, and the correspondence*

$$x \longmapsto S(x) = \{P \in L^*: x \in P\}$$

is an isomorphism from L *onto the lattice* L** *of all compact open sets of* L*. *Conversely, let* X *be a Stone space and let* X* *be the lattice of all compact open* *sets of* X. *Then the map*

$$x \longmapsto \{a \in X^*: x \in a\}$$

is a homeomorphism from X *onto* X^{**}, *the set of all prime filters of* X^* *with the weak topology.*

2.3. THEOREM. (Nerode [11]). *Let* L *and* L' *be distributive lattices with* 0 *and* 1, *and* $h:L \longrightarrow L'$ *a homomorphism with* $h(0) = 0$ *and* $h(1) = 1$. *If for any* P' *in* L'^* *we define* $h^*(P') = h^{-1}(P')$, *the* h^* *is a strongly continuous map from* L'^* *into* L^*, *and the correspondence* $h \longmapsto h^*$ *establishes a bijection between the set of all homomorphisms from* L *into* L' *preserving* 0 *and* 1, *and the set of all strongly continuous functions from* L'^* *into* L^*.

From the above two theorems, it follows at once that:

2.4. COROLLARY. *Let* $\mathcal{D}$ *be the category of distributive lattices with* 0 *and* 1, *and homomorphism preserving* 0 *and* 1. *Then the category* $\mathcal{D}^*$ *of Stone spaces and strongly continuous functions is naturally equivalent to the dual category of* $\mathcal{D}$.

Therefore, if A is a n-valued Lukasiewicz algebra, and if A^* denotes the set of all prime filters of A endowed with the weak topology, it follows that A^* is a Stone space on which there are defined the n functions $g_o, g_1, \ldots, g_{n-1}$. Some topological properties of these functions are established in the following:

2.5. LEMMA. *The functions* $g_o, g_1, \ldots, g_{n-1}$ *have the following properties:*

A5) *The complementary set of the image by* g_o *of a compact open set is compact open.*

A6) *The functions* g_i, $i=1,\dots,n-1$ *are strongly continuous.*

A7) *If* X *is a compact open set of* A*, *then* $g_1^{-1}(X) \subseteq \dots \subseteq$ $\subseteq g_{n-1}^{-1}(X)$.

PROOF. Let X be a compact open subset of A*. From theorem 2.2 it follows that there is an x in A such that $X = S(x)$, and from the results of [1] we get $C\, g_o(X) = C\, g_o(S(x)) = S(-x)$. Using again Theorem 2.2, we see that $S(-x)$ is a compact open set of A*, which proves A5). A6) follows at once from Theorem 2.3, because $g_i(X) = s_i^{-1}(X) = s_i^*(X)$, and the s_i are lattice homomorphisms from A into A preserving 0 and 1. To prove A7), observe that $S(s_1(x)) \subseteq \dots \subseteq S(s_{n-1}x)$, and that $S(s_i x) = g_i^{-1}(X)$, $i=1,\dots,n-1$.

2.6. DEFINITIONS. *The prime spectrum of a n-valued Lukasiewicz algebra* A *is the system* $(A^*, g_o, g_1, \dots, g_{n-1})$ *where* A* *is the set of all prime filters of* A *endowed with the weak topology, and* g_i $(i=0,1,\dots,n-1)$ *are the functions from* A* *into* A* *defined by formulae* (1) *and* (2) *respectively.*

2.7. DEFINITIONS. A *Lukasiewicz space of order* n *is a system* $(E, g_o, g_1, \dots, g_{n-1})$ *such that* E *is a Stone space and* g_i $(0 \leqslant i \leqslant n-1)$ *are* n *functions from* E *into* E *fulfilling the properties* A1) - A4) *of Lemma 1.1 and the properties* A5) - A7) *of Lemma 2.5.*

If $(E, g_o, \dots, g_{n-1})$ *and* $(E', g'_o, \dots, g'_{n-1})$ *are Lukasiewicz* *spaces of order* n*, we say that a strongly continuous function* $f:E \longrightarrow E'$ *is L-continuous if it satisfies the condition:*

$$\text{LC)} \quad f\, g_i = g'_i\, f \quad \textit{for } i=0,1,\dots,n-1.$$

We say that $(E,g_o,\dots,g_{n-1})$ *and* $(E',g'_o,\dots,g'_{n-1})$ *are isomorphic if there are L-continuous functions* $f:E \longrightarrow E'$ *and* $g:E' \longrightarrow E$ *such that* gf = *identity on* E *and* fg = *identity on* E'.

Note that since the strongly continuous functions are continuous, it follows that $(E,g_o,\dots,g_{n-1})$ and $(E',g'_o,\dots,g'_{n-1})$ are isomorphic if and only if there is an homeomorphism $f: E \longmapsto E'$ onto, with the property LC).

2.8. LEMMA. *Let* $(E,g_o,\dots,g_{n-1})$ *be a Lukasiewicz space of order* n *and let* E* *be the set of all compact open subsets of* E. *If we define the operations* $-$ *and* s_i *for* $i=1,\dots,n-1$ *on* 2^E *by the formulae* (3) *and* (4) *respectively, then the system* $(E^*,E,\cup,\cap,-,s_1,\dots,s_{n-1})$ *is a n-valued Lukasiewicz algebra, called the Lukasiewicz algebra associated with the Lukasiewicz space* $(E,g_o,\dots,g_{n-1})$.

PROOF. From Theorem 2.2 it follows that $(E^*,E,\cup,\cap)$ is a distributive lattice with unit E. On the other hand, properties A5) and A6) imply that E* is closed under the operations $-$ and s_i $(1 \leqslant i \leqslant n-1)$; and A7) implies that E* satisfies the axiom L5). An application of Lemma 1.2 completes the proof.

2.9. THEOREM. *If* $(A^*,g_o,\dots,g_{n-1})$ *is the prime spectrum* *of the n-valued Lukasiewicz algebra* A*, then it is a Lukasiewicz space of order* n*, and* A *is isomorphic to the Lukasiewicz algebra associated to* A*, A**.

PROOF. From lemmas 1.1 and 2.5 it follows that $(A^*, g_o, \ldots, g_{n-1})$ is a Lukasiewicz space of order n, and the proof of Theorem 1.5 can be applied now to show that A and A** are isomorphic.

2.10. THEOREM. *Any Lukasiewicz space of order* n*, is isomorphic to the prime spectrum of its associated Lukasiewicz algebra.*

PROOF. Let $(E, g_o, \ldots, g_{n-1})$ be a Lukasiewicz space of order n, E* its associated Lukasiewicz algebra and $(E^{**}, g'_o, \ldots, g'_{n-}$ the prime stectrum of E*. From theorem 2.2 it follows that the topological spaces E and E** are isomorphic, and that the isomorphism $f\colon E \longrightarrow E^{**}$ is given by $f(x) = P_x = \{a \in E^* : x \in a\}$. In order to complete the proof we have to show that $f(g_i(x)) = g'_i(f(x))$ for any x in E and i=0,... ...,n-1; but this can be easily done taking into account the definitions of the operations - and s_i in E* and the functions g'_i $(1 \leqslant i \leqslant n-1)$ on E**.

The next two lemmas are easy consequences of the given definitions, and so their proofs will be omited.

2.11. LEMMA. *Let* A *and* A' *be n-valued Lukasiewicz algebra, with prime spectra* A* *and* A'* *respectively, and* $h\colon A \longrightarrow A'$ *a homomorphism. If for any prime filter* P' *of* A' *we define* $h^*(P') = h^{-1}(P')$*, then* h* *is a L-continuous function from* A'* *into* A*.

2.12. LEMMA. *Let* $(E, g_o, \ldots, g_{n-1})$ *and* $(E', g'_o, \ldots, g'_{n-1})$

be Lukasiewicz spaces of order n, *with associated Lukasiewicz algebras* E^* *and* E'^* *respectively, and* $f: E \longrightarrow E'$ *an L-continuous function. If for any compact open subset* a' *of E' we define* $f^*(a') = f^{-1}(a')$, *then* f^* *is a homomorphism from* E'^* *into* E^*.

Summing up the above results we obtain:

2.13. THEOREM. *Let* L_n *denotes the category of n-valued Lukasiewicz algebras and homomorphisms. The category* L_n^* *of Lukasiewicz spaces of order* n *and L-continuous functions is naturally equivalent to the dual category of* L_n.

We are going to close this section with some remarks on the structure of the Lukasiewicz spaces.

In the remarks following Lemma 1.1 we have pointed out that if we denote by f_i the correspondence $P_i^* \longmapsto P^*$, then the f_i $(1 \leq i \leq n-1)$ are injections from the set M of ultrafilters of the Boolean algebra B(A) into the set E of prime filters of A, and that if $E_i = f_i(M)$, then $E = \bigcup_{i=1}^{n-1} E_i$. Let $B(A)^*$ denotes the set M endowed with the weak topology, and A_i^* the set E_i considered as a subspace of A^*, i.e., the set E_i with the relative topology of the weak topology on E.

Since $f_i(P^*) = s_i^{-1}(P^*)$ and the s_i are lattice homomorphism from A into B(A), it follows from Theorem 2.3 that the f_i are strongly continuous functions from $B(A)^*$ into A^*, and therefore, *continuous bijections* from $B(A)^*$ onto A_i^*. Actually they are homeomorphism from $B(A)^*$ onto A_i^*. For, if u is a compact open set in $B(A)^*$, then there is an b in B(A) such that $u = S'(b) = \{P^* \in M: b \in P^*\}$, and since $B(A) \subseteq A$, we can define $v = S(b) = \{P \in E: b \in P\}$. v is then a compact open set in A^*, and it is not hard to see that $f_i(u) = v \cap A_i^*$. This shows the functions f_i are also open in the topology of A_i^*.

So, we have proved the following:

2.14. PROPOSITION. *The prime spectrum of a n-valued Lukasiewicz algebra* A *is the set theoretical union of* n-1 *subspaces, each of them homeomorphic to the Stone space of the Boolean algebra* B(A).

It is easy to construct examples in which the sets A_i^* are neither open nor closed in A^*.

2.15. PROPOSITION. *Any compact open set* v *in* A^* *can be written in the form:*

(5) $v = \bigcup_{i=1}^{n-1} f_i(u_i)$, *where the* u_i *are closed open sets in* $B(A)^*$ *such that* $u_1 \subseteq u_2 \subseteq \dots \subseteq u_{n-1}$

PROOF. Taking into account that if $P = P_i^*$ then the conditions $x \in P$ and $s_i x \in P^*$ are equivalent, it follows at once that:

$$\text{(i)} \qquad S(x) \cap A_i^* = S(s_i(x)) \cap A_i^* .$$

If v is a compact open set in A^*, by theorem 2.2 there is an x in A such that $v = S(x)$. The above proposition together with (i) imply:

$$v = v \cap A^* = v \cap \bigcup_{i=1}^{n-1} A_i^* = \bigcup_{i=1}^{n-1} (S(s_i x) \cap A_i^*)$$

If S' is the Stone transformation from B(A) into the closed open sets of $B(A)^*$, we have $f_i(S'(s_i x)) = S(s_i x) \cap A_i^*$, hence setting $u_i = S'(s_i x)$, the u_i are closed open sets in $B(A)^*$, and by L5) it follows that $u_1 \subseteq \dots \subseteq u_{n-1}$.

If is possible to construct examples showing that the representation (5) is not unique, and that the sets of the form (5) are not necessarily compact open in A*. In connection with this remarks we have:

2.16. LEMMA. *If* $A_i^* \cap A_j^* = \emptyset$ *for* $i \neq j$, *then the representation* (5) *is unique. More precisely, if* $v = S(x)$, *then* $u_i = S'(s_i x)$, *then* S' *is as in the proof of Proposition* 2.15.

PROOF. Suppose that $S(x) = \bigcup_{i=1}^{n-1} f_i(u_i)$, where the u_i are closed open sets in $B(A)^*$ and $u_1 \subseteq \ldots \subseteq u_{n-1}$ and let $u_i = S'(b_i)$, where b_i is in $B(A)$. It is clear that $b_1 \leqslant \ldots \leqslant b_{n-1}$. Since $f_i(S'(b_i)) = S(b_i) \cap A_i^*$, it follows that for any j, $1 \leqslant j \leqslant n-1$, we have:

$$s_j S(x) = g_j^{-1}(S(x)) = g_j^{-1}\left(\bigcup_{i=1}^{n-1}(S(b_i) \cap A_i^*)\right) =$$

$$= \bigcup_{i=1}^{n-1}(g_j^{-1}(S(b_i) \cap g_j^{-1}(A_i^*))$$

and since $g_j^{-1}(A_i^*) = \emptyset$ if $i \neq j$ and $g_j^{-1}(A_j^*) = A^*$, the above equations imply $s_j S(x) = g_j^{-1}(S(b_j) = s_j S(b_j) = S(s_j b_j) = S(b_j)$. Hence $s_j S(b_j) = S(s_j x) = S(b_j)$, and since S is an isomorphism, it follows that $s_j x = b_j$. Therefore, $u_j = S'(b_j) = S'(s_j x)$, which completes the proof.

3. POST ALGEBRAS.

In [4], Theorem 8.6 we have characterized Post algebras of order n as n-valued Lukasiewicz algebras with n-2 elements $e_1, \ldots, e_{n-2}$ fulfilling the equations:

$$(6) \qquad s_i\, e_j = \begin{cases} 0 \text{ if } i+j < n \\ 1 \text{ if } i+j \geqslant n \end{cases} \qquad i=1,\dots,n-1\ ,\ j=1,\dots,n-2.$$

From the remarks following Lemma 1.1 and the Theorem 8.8 of [4] it follows that for a Post algebra of order n A, the sets A_i^* $(1 \leqslant i \leqslant n-1)$ are mutually disjoint, and lemma 2.16 and equations (6) imply that $S(e_j) = \bigcup_{i=1}^{j} A_i^* = F_j$, $j=1,\dots,n-2$. Therefore, the sets F_j are basic open sets in A^*.

3.1. LEMMA. *If* A *is a Post algebra of order* n, *then any set of the form* (5) *is compact open in* A^*.

PROOF. If v is represented by equation (5), setting $u_i = S'(b_i)$ as in the proof of lemma 2.16, we get that $v = \bigcup_{i=1}^{n-1} (S(b_i) \cap A_i^*)$. Since $S(b_i)$ is a closed open set in A^* (because b_i is in the Boolean algebra $B(A)$); $S(b_i) \cap A_i^*$ is a closed set in A_i^*, therefore a compact subset of A_i^*. Hence, v is compact in A^*. Moreover, from $S(b_1) \subseteq \dots \subseteq S(b_{n-1})$ it follows at once that $v = \bigcup_{i=1}^{n-1} (S(b_i) \cap F_i)$, where $F_{n-1} = A^*$; and since the sets F_i are open in A^*, $S(b_i) \cap F_i$ is open in A^* for $i=1,\dots,n-1$, and so v is open.

Summing up the above results, we can establish the following:

3.2. THEOREM. *Let* A *be a Post algebra of order* n, *and let* A^* *be the set of prime filters of* A *endowed with the weak topology. Then there exists a totally disconnected compact Hausdorff space* M *and* n-1 *homeomorphisms*

$f_i\colon M \longrightarrow A^*$ $(1 \leq i \leq n-1)$ *such that:*

1) $A^* = \bigcup_{i=1}^{n-1} f_i(M)$.

2) $f_i(M) \cap f_j(M) = \emptyset$ *if* $i \neq j$.

3) *The compact open sets form an open basis for the topology of* A^**, and are just the sets of the form* $v = \bigcup_{i=1}^{n-1} f_i(u_i)$*, where the* u_i *are closed open sets in* M *and* $u_1 \subseteq \ldots \subseteq u_{n-1}$. *Moreover,* M *is homeomorphic to the Stone space of the Boolean algebra* $B(A)$*, and* A *is isomorphic with the Post algebra of all compact open sets of* A^*.

Following the same lines of T.Traczyk [14] and P.Dwinger [6] it is possible to prove that the properties enounced in Theorem 3.2 characterize the set of prime filters of Post algebras with the weak topology. It is worthwhile to point out, however, that these authors considered the set of prime filters of A endowed with the topology having the sets of the form $\bigcup_{i=1}^{n-1} f_i(u)$, where u is a closed open set in $B(A)^*$ as an open basis, and this topology is actually weaker than the Stone topology on A^*.

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