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ON SCHEDULING MULTIPROCESSING SYSTEMS

IN A HARD REAL-TIME ENVIRONMENT

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1. INTRODUCTION

Our purpose is to answer a question that arises when the problem of scheduling a set of tasks that must be performed within certain constraints of time is studied. It was proposed to us by Ing. Jorge Santos, of the Laboratorio de Sistemas Digitales, Departamento de Ingeniería Eléctrica, Universidad Nacional del Sur, as an open matter, in october 1989.

The use of computers for control and monitoring of industrial processes and the requirement of an efficient utilization of the medium make necessary a careful scheduling of the operations.

The problem is the following: There is a set of sources $\eta_1, \ldots, \eta_n$ (nodes) that emit electric signals to activate n mechanisms that perform tasks $\tau_1, \ldots, \tau_n$, which by simplicity will be assumed to require the same time to be fulfilled. Each source η_i emits its signal at regular periods of width T_i and the corresponding task τ_i must be performed exactly once before η_i sends its next signal and not simultaneously with any other task.

As an example one can think of an automatic industrial process where a set of operations must be regularly performed, each at a time and no one can be missed, although their frequencies may not be the same and the beginning of the execution of each task can be delayed within certain limits.

We shall then consider a set of n nodes that periodically request to transmit a message, on the following conditions:

1 - At the starting point of the process all the nodes request simultaneously to transmit their messages.

2 - Each node requests to transmit its message at regular intervals. The time elapsed between two consecutive requests of a node is called the crisis time of this node.

3 - When a node requests to transmit, the corresponding message must be delivered exactly once before its next request. It is said that the system operates in a hard real-time environment (when some misses are allowed it is called a soft real-time environment).

4 - All the messages require the same time to be fulfilled. This time, that will be called the slot-time, also includes the time of transmission of the signal from the sending node as well as some additional necessary for the computer to determine which node will be the next to accede to the medium, if there is any ready to transmit.

5 - The crisis times of the nodes are integer multiples of the slot-time.

6 - The starting of any message is synchronous with the beginning of a slot.

7 - The transmission of a message is not interrupted until it is finished. Because of this condition the system is said to be not preemptive.

8 - The medium does not remain idle if there is any node requesting it.

When a message is not delivered before its deadline it is said that crisis occurs. The problem is to schedule the system avoiding crisis.

Following [1] we shall say that a priority discipline PD is given for a set of nodes if at the beginning of each slot a linear ordering relation is defined on the set of nodes. The ordered set is called the priority-stack; at the beginning of each slot the medium is assigned to the node that has the higher position in the stack among those with messages generated. Most of the deterministic (i.e. non-random) priority disciplines suitable for real-time systems are variations or combinations of three main types of priority assignments:

Round Robin, Fixed Priority and Least Time to Crisis.

In a Round Robin discipline (RRD), the priority stack for a given slot is obtained from the preceding one transferring the node at the top to the bottom. It is a dynamic discipline in the sense that the stack is a function of time.

In a Fixed Priority discipline (FPD) the stack is assembled once for all; it is therefore a static discipline. For instance, the stack could be ordered by increasing crisis times.

In a Least Time to Crisis discipline (LTCD), the stack at each slot is induced by the time available to each node before reaching crisis. If there are several nodes with the same time left, these nodes are ordered according to a certain linear ordering on the set of nodes chosen beforehand.

Given a set of n nodes $\eta_1, \dots, \eta_n$ and chosen any priority discipline PD for it, eventually a random one, we shall say that the system is crisis-free or compatible for that PD if none crisis occurs. Otherwise it will be said incompatible. A crisis-free system will be said saturated if every slot is engaged in the transmission of a message and non saturated if not. We shall say that a slot is full or empty according there is or there is not a message being transmitted at it.

Taking into account the assumptions already made, we can regard the slot-time as the time unit, time as always measured by an integer number of units and the time space (the slot space) represented by the set N of natural numbers, and so we do throughout this paper. Sometimes we shall say instant meaning slot.

The interval [a,b] will denote the set $\{x \in N: a \leq x \leq b\}$. The case $a = b$ is not ruled out.

We consider here the following questions:

(A) Decide if a system of nodes is crisis-free for a given PD.

(B) Given a crisis-free non saturated system of n nodes for a PD, find conditions to enlarge it to a crisis-free system adding one more node.

An answer to (A) was given in [2] in the case the chosen discipline is a LTCD, and in [1] in the RRD case. The remaining question is to answer it for a FPD.

Liu and Layland proved in [2] that a system of n nodes is crisis-free for the LTCD if and only if $\sum_{i=1}^{n} \frac{1}{T_i} \leq 1$, being $T_1, \ldots, T_n$ the crisis times of the nodes. But this condition does not characterize a crisis-free system for other types of priority disciplines, since although any compatible system for a PD verifies the above relation, the converse is not necessarily true. In particular, this is the case for a RRD or a FPD.

As an example, consider a set of four nodes $\eta_1, \eta_2, \eta_3, \eta_4$ with crisis times $T_1 = 5$, $T_2 = 4$, $T_3 = 7$, $T_4 = 3$. Then $\sum_{i=1}^{4} \frac{1}{T_i} < 1$. It is clear that this set of nodes is incompatible for a RRD. Choose now the FPD that orders the set of nodes according to the usual increasing ordering of their crisis times, that is $\{\eta_4, \eta_2, \eta_1, \eta_3\}$. This FPD is called the rate-monotonic FPD. It can be easily seen that the system is incompatible for this FPD. Since in [2] it was also proved that if a set of nodes is crisis-free for a FPD then it is crisis-free for the rate-monotonic FPD, it follows that the set of nodes in our example is incompatible for any FPD.

In [1] Santos, Orozco and Alimenti gave the following criterion: A set of n nodes is crisis-free for a RRD if and only if n is not greater than the minimum of their crisis times, as it is easy to see.

In this paper we shall answer (A) for a FPD, and (B) for any PD, choosing a suitable PD for the enlarged system.

2. RESULTS

Observe that if a node has crisis time T then it requests to transmit its message at the instants 1, T+1, 2T+1,......, kT+1,..... , k ε **N** . So from the beginning up to an instant x it makes $[\frac{x}{T}]$ requests, where [y] denotes the smallest integer larger than or equal to y.

Therefore the number of the requests that makes a system of n nodes with crisis times $T_1, T_2, \ldots, T_n$ up to an instant x is equal to

$[\frac{x}{T_1}] + [\frac{x}{T_2}] + \ldots + [\frac{x}{T_n}]$.

We introduce the following function

<u>DEFINITION</u>. Given n nodes with crisis times $T_1, \ldots, T_n$ let

$f_{(T_1,\ldots,T_n)} : \mathbf{N} \to \mathbf{N}$ be the function associated with them defined as follows

$$f_{(T_1,\ldots,T_n)}(x) = \sum_{i=1}^{n} [\frac{x}{T_i}] \quad , \quad x \in \mathbf{N} .$$

For simplicity in notation we shall henceforth delete the subscript and write simply f(x) if no confusion is possible.

f is a non decreasing function of x and has the following property that will be used further on:

<u>LEMMA</u>. $f(x+y) = f(x) + f(y) - j$ with $0 \leqq j \leqq n$, $x,y \in \mathbf{N}$.

Proof. It is easy to see that

$$[\frac{a+b}{c}] = \begin{cases} [\frac{a}{c}] + [\frac{b}{c}] \\ \text{or} \\ [\frac{a}{c}] + [\frac{b}{c}] - 1 \end{cases}$$

Hence

$$[\frac{a+b}{c}] = [\frac{a}{c}] + [\frac{b}{c}] - j \quad \text{with} \quad 0 \leq j \leq 1.$$

Then

$$f(x+y) = \sum_{i=1}^{n} [\frac{x+y}{T_i}] = \sum_{i=1}^{n} ([\frac{x}{T_i}] + [\frac{y}{T_i}] - j_i) =$$

$$= \sum_{i=1}^{n} [\frac{x}{T_i}] + \sum_{i=1}^{n} [\frac{y}{T_i}] - \sum_{i=1}^{n} j_i = f(x) + f(y) - j \quad \text{where}$$

$j = \sum_{i=1}^{n} j_i$. As $0 \leq j_i \leq 1$ for all index $i = 1,2,\ldots,n$ is $0 \leq j \leq n$.

This proves the lemma.

As we pointed out

(1) $f(x)$ = number of the requests that the system of n nodes makes in the time interval $[1,x]$.

Then

(2) $f(x+1) - f(x)$ = number of the requests that occur at the instant $x+1$.

Therefore, for a crisis-free system, if $f(x+1) - f(x) > 0$ the interval $[x+1 \, , \, x+(f(x+1)-f(x))]$ is full, that is, all its slots are occupied by transmissions; if $f(x+1) = f(x)$ then the slot $x+1$ can be full or empty. It is also clear that if $f(x) \geq x$ then the slot x is full; if $f(x) < x$ then the slot x can be full or empty.

In general

(3) $f(y) - f(x)$ = number of the requests that occur in the interval $[x+1,y]$.

Notice that since at the starting point all the nodes are simultaneously activated, this situation arises again for the first time at the instant M+1, being M the least common multiple of their crisis times. Therefore to study the behaviour of the system for a PD it suffices to analyse it in the time interval [1,M].

If the system is crisis-free then all the requests produced in [1,M] can be satisfied within this interval and in consequence $f(M) \leqq M$, and the difference $M - f(M)$ gives the number of slots left empty by the system in [1,M].

Then, <u>a crisis-free system is saturated if and only if $f(M) = M$</u>.

Since M is a multiple of each crisis time we have $[\frac{M}{T_i}] = \frac{M}{T_i}$, for $i = 1,\dots,n$, and thus

$$f(M) = \sum_{i=1}^{n} [\frac{M}{T_i}] = \sum_{i=1}^{n} \frac{M}{T_i} = M \sum_{i=1}^{n} \frac{1}{T_i} .$$

Therefore

$$f(M) \leqq M \qquad \text{if and only if} \qquad \sum_{i=1}^{n} \frac{1}{T_i} \leqq 1 .$$

So the above criterion for a crisis-free system to be saturated is equivalent to the following:

<u>A crisis-free system is saturated if and only if $\sum_{i=1}^{n} \frac{1}{T_i} = 1$.</u>

Now we are going to compute the set of slots left empty by a crisis-free non saturated system. Being the system non saturated is $f(M) < M$ and $s = M - f(M)$ is the number of empty slots in the interval [1,M]. Let them be $e_1,\dots,e_s$. Then the slots left empty by the system in $[1, +\infty)$ are $e_1 + kM,\dots, e_s + kM$, for $k = 0,1,2,\dots$. Consider the set of empty slots with the usual ordering.

THEOREM 1. Given a crisis-free non saturated system for a PD, the i-th empty slot e_i left by the system is the least $x \in [1, +\infty)$ such that $f(x) = x - i$, for all $i \in N$.

Proof. Let e_i be the i-th empty slot. As in the interval $[1, e_i-1]$ there are exactly i-1 empty slots, it must be $f(e_i-1) \geqq e_i - 1 - (i - 1) = e_i - i$. The inequality $f(e_i-1) > e_i - i$ would imply that the slot e_i is full, so that $f(e_i-1) = e_i - i$. Moreover, since none request occurs at e_i, it is $f(e_i-1) = f(e_i)$, then $f(e_i) = e_i - i$. It remains to show that e_i is the least $x \in [1, +\infty)$ such that $f(x) = x - i$. For this it suffices to notice that if $x < e_i$ then the number of empty slots in the interval $[1,x]$ is at most i-1, which implies that $f(x) \geqq x - (i-1)$, that is $f(x) > x - i$ for all $x < e_i$. This ends the proof.

COROLLARY. Any two PD that make a given set of nodes crisis-free leave empty the same set of slots.

Proof. As the value of the function f(x) does not depend on the ordering of the set of nodes but just on their crisis times, the corollary clearly follows from the proof of theorem 1.

THEOREM 2. Given a crisis-free non saturated system for a PD, let $\{e_1, e_2, ...\}$ be the set of slots left empty, with the usual ordering. Then

a) $e_{i+1} - e_i \leqq e_1$ for all i.

b) The slot M is an empty one, where M is the least common multiple of the crisis times of the system.

Proof. To prove a) it will suffice to show that $e_{i+1} \in [e_i+1, e_i+ e_1]$ for all i. The interval $[e_i+1, e_i+e_1]$ has e_1 slots and there is none request previous to the instant e_i+1 waiting to be fulfilled since e_i is an empty slot. We compute the number r of requests that occur in this interval using (3) and the Lemma:

$$r = f(e_i+e_1) - f(e_i) = f(e_i) + f(e_1) - j - f(e_i)$$

$$r = f(e_1) - j \quad \text{with} \quad 0 \leqq j \leqq n\,, \quad n = \text{number of nodes in the system.}$$

As $f(e_1) = e_1 - 1$ we have

$$r = e_1 - 1 - j < e_1\,.$$

Since the number of requests is smaller than the number of available slots in the interval, there must be at least one empty slot, hence $e_{i+1} \in [e_i+1, e_i+e_1]$ as was to be shown.

To prove b), let $s = M - f(M)$. We already know that s is the number of empty slots $e_1, \ldots, e_s$ in the interval [1,M] and that $e_{s+1} = e_1 + M$. By a) is $e_{s+1} - e_s \leqq e_1$ which implies $M \leqq e_s$. As $e_s \leqq M$ it follows that $e_s = M$.

We now consider the question (B) put before: Given a crisis-free non saturated system with n nodes for a PD, enlarge it to a crisis-free system with one more node. To do this the PD chosen for the new system will be the one obtained extending the PD defined on the given set of n nodes such as the added node has the last priority at each slot.

<u>THEOREM 3</u>. Let $S = \{\eta_1, \ldots, \eta_n\}$ be a crisis-free non saturated system for a PD and consider a system $S' = \{\eta_1, \ldots, \eta_n, \eta_{n+1}\}$ with the PD that is obtained adding the new node to the PD for S as the last choice at each slot.

Then S' is crisis-free if and only if $T_{n+1} \geqq e_1$, where e_1 is the first empty slot left by S.

Proof. Suppose that $T_{n+1} \geqq e_1$. The new node generates its message at the instants $1, T_{n+1} + 1, 2T_{n+1} + 1, \ldots, kT_{n+1} + 1, \ldots, k = 0,1,2,\ldots$.
To prove that the system S' is crisis-free it suffices to show that each request of the new node finds an available slot in the corresponding interval $[kT_{n+1} + 1, (k+1)T_{n+1}]$, being $kT_{n+1} + 1$ the instant at which the given request occurs. But all these intervals have length T_{n+1} and since $T_{n+1} \geqq e_1$, from Theorem 2, a) follows that in each of them there is at least one of the slots left empty by the system S, which proves that S' is crisis-free.

Conversely, suppose that S' is crisis-free. As by hypothesis the new node has the last priority at each slot, it is obvious that it must be $T_{n+1} \geqq e_1$.

THEOREM 4. A set of n nodes with n not greater than the minimum of their crisis times is crisis-free for any fixed priority discipline FPD.

Proof. Let $\{\eta_1, \ldots, \eta_n\}$ be a set of nodes ordered according to a given FPD, $n \leqq \min\{T_1, \ldots, T_n\}$.

Proceeding by induction on n, if n = 1 the system is obviously crisis-free.
Let n > 1 and suppose the theorem true for any set of n-1 nodes satisfying the condition on the number of nodes.
Then the subsystem $\{\eta_1, \ldots, \eta_{n-1}\}$ is crisis-free since $n-1 < n \leqq$ $\leqq \min\{T_1, \ldots, T_n\} \leqq \min\{T_1, \ldots, T_{n-1}\}$. The first slot left empty by it is $e_1 = n$, as the crisis times of the n-1 first nodes are greater than or equal to n. Since $T_n \geqq n$, from Theorem 3 follows that the system $\{\eta_1, \ldots, \eta_{n-1}, \eta_n\}$ is crisis-free for the given FPD.

Next we shall consider the question (A) and answer it in the FPD case, developing a method for deciding if a set of nodes is crisis-free for a given FPD.

In what follows to say that $\{\eta_1, \ldots, \eta_n\}$ is a system with a given FPD will imply that the set of nodes appears ordered according to it. Each subset $\{\eta_1, \ldots, \eta_k\}$, $1 \leq k \leq n$, with the given FPD restricted to it will be called a subsystem of $\{\eta_1, \ldots, \eta_n\}$.

It is clear that a system $\{\eta_1, \ldots, \eta_n\}$ is crisis-free for a FPD if and only if every subsystem $\{\eta_1, \ldots, \eta_k\}$ is crisis-free, $k = 1, \ldots, n$.

To simplify notation, let S be the given system, and for $k = 1, \ldots, n$ denote S_k the subsystem $\{\eta_1, \ldots, \eta_k\}$, f_k the function f associated with S_k, M_k the least common multiple of the crisis times $T_1, \ldots, T_k$ and $\Sigma(k)$ the number $\sum_{i=1}^{k} \frac{1}{T_i}$.

3. METHOD FOR DECIDING IF A SYSTEM $\{\eta_1, \ldots, \eta_n\}$ WITH A GIVEN FPD IS CRISIS-FREE.

Let $m = \min \{T_1, \ldots, T_n\}$.

CASE I. $n \leq m$

Then by theorem 4 S is crisis-free.

If $n < m$, S is non saturated.

If $n = m$ then S is saturated if and only if $\Sigma(n) = 1$, that is, if and only if $T_1 = \ldots = T_n$.

CASE II. $n > m$

By Theorem 4 the subsystem $S_m = \{\eta_1, \ldots, \eta_m\}$ is crisis-free.

If $\Sigma(m) = 1$ then S_m is saturated and as $n > m$, S is incompatible and we finished.

If $\Sigma(m) < 1$, S_m leaves empty $M_m - f_m(M_m)$ slots in the interval $[1, M_m]$. We look for the first of them $e_{(m)1}$, which by Theorem 1 and our assumptions is the least $x \in [m+1, M_m]$ such that $f_m(x) = x-1$. Then, by Theorem 3 the subsystem $S_{m+1} = \{\eta_1, \ldots, \eta_m, \eta_{m+1}\}$ is crisis-free if and only if $T_{m+1} \geqq e_{(m)1}$.

If $n = m+1$, $S_{m+1} = S$ and we have finished.

If $n > m+1$, S_{m+1} incompatible implies S incompatible and we finished. If S_{m+1} is crisis-free, as $n > m+1$, we next repit the process on the subsystem S_{m+1} to analyse the compatibility of $S_{m+2} = \{\eta_1, \ldots, \eta_{m+1}, \eta_{m+2}\}$.

Now we give a more systematic description of the procedure, useful for computational purposes. With the above notation, THE METHOD CONSISTS IN PERFORMING THE FOLLOWING STEPS:

(i) If $n \leqq m$ then S is crisis-free and we finished.

(ii) If $n > m$ then the subsystem S_m is crisis-free. Apply (iii), being $e_{(m-1)1} = m$.

(iii) If the subsystem S_k, with $m \leqq k \leqq n-1$, is crisis-free compute $\Sigma(k)$. If $\Sigma(k) = 1$ then S_k is saturated and as $k < n$ it follows that S is incompatible and we finished.

If $\Sigma(k) < 1$, look for $e_{(k)1}$ = the least $x > e_{(k-1)1}$ such that $f_k(x) = x-1$.

If $e_{(k)1} > T_{k+1}$ then S is incompatible and we finished.

If $e_{(k)1} \leqq T_{k+1}$ then S_{k+1} is crisis-free. If $k+1 = n$, S is crisis-free and we finished. If $k+1 < n$ apply (iii) to S_{k+1}.

REMARKS.

1) If a system $\{\eta_1, \ldots, \eta_n\}$ with a given FPD is incompatible, the above method determines the smallest k such that the system $\{\eta_1, \ldots, \eta_k\}$ is incompatible, which eventually would allow to replace the crisis time T_k by another more convenient.

2) Given a crisis-free non saturated system $\{\eta_1, \ldots, \eta_n\}$ the method shows how to find the least crisis times $T_{n+1}, T_{n+2}, \ldots$ to extend the given system to compatible ones.

3) In [2] it was proved that if a set of nodes is crisis-free for a FPD then it is crisis-free for the rate-monotonic FPD.
Therefore to know if it is possible for a given set of nodes to be crisis-free for some FPD, it is sufficient to find out if it is so for the rate-monotonic FPD.

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