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SEMI-HEYTING ALGEBRAS: AN ABSTRACTION FROM HEYTING ALGEBRAS

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Dedicated to Professor A. Monteiro

ABSTRACT. The purpose of this paper is to define and investigate a new (equational) class of algebras called "Semi-Heyting Algebras" as an abstraction from Heyting algebras. We show that semi-Heyting algebras are distributive pseudocomplemented lattices, the congruences on these algebras are determined by filters, and the variety $\mathscr{SH}$ of semi-Heyting algebras is arithmetical, thus extending the corresponding results of Heyting algebras. Using these results, we characterize the directly indecomposables, simples, and subdirectly irreducibles in $\mathscr{SH}$. We also show that, in $\mathscr{SH}$, there are 2 two-element algebras—which turn out to be primal, 10 three-element algebras and 160 four-element algebras. Also, equational bases for the 2-element and 3-element semi-Heyting algebras, as well as several new axiom systems for Heyting algebras, are given. Some important subvarieties of $\mathscr{SH}$ are defined. We also present three generalizations of semi-Heyting algebras and point out which of the results of this paper remain true in these generalizations. The paper concludes with some open problems, which, we hope, will stimulate further research.

1. INTRODUCTION

The purpose of this paper is to define and investigate a new (equational) class of algebras, which we call *"Semi-Heyting Algebras"*, as an abstraction from Heyting algebras. We were led to the discovery of these algebras in 1983-84, as a result of our research that went into [29]. Some of the early results were announced in [30].

A closer look at the proofs of results proved in [29] led us to the following rather interesting observation: The arguments used in [29], for the most part, used only the following well known properties of Heyting algebras:

(1) They are pseudocomplemented,
(2) They are distributive, and
(3) Congruences on them are determined by filters.

This observation led us to the following conjecture.

CONJECTURE: There exists a variety $\mathscr{V}$ of algebras (of the same type as that of Heyting algebras) such that (a) $\mathscr{V}$ possesses all three properties mentioned above and includes Heyting algebras, and (b) almost all of the results of [29] generalize to a variety $\mathscr{V}^e$, whose type is the expansion of the type of $\mathscr{V}$ obtained by adding the dual pseudocomplementation.

We show in this paper that part (a) of the above conjecture is true if we take $\mathscr{V}$ to be the variety $\mathscr{SH}$ of semi-Heyting algebras. Part (b) is proven in the paper [34], also with

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$\mathscr{SH}$ as $\mathscr{V}$. We also show that semi-Heyting algebras share with Heyting algebras some rather strong properties, besides the three mentioned earlier. For example, every interval in a semi-Heyting algebra is also pseudocomplemented, and the variety of semi-Heyting algebras is arithmetical.

The contents of the paper are summarized as follows: In Section 2 we define semi-Heyting algebras and present some basic arithmetical properties, along with several alternate definitions for them. Section 3 deals with the structure of the underlying lattice of a semi-Heyting algebra. Besides proving the first two properties mentioned above, Section 3 also presents several additional properties. In Section 4 we show that there are ten non-isomorphic semi-Heyting algebras on a 3-element chain, only one of which, of course, is a Heyting algebra. The 3-element Heyting algebra is well known as the 3-valued intuitionistic logic—the model (matrix) that provides the semantics for the 3-valued intuitionistic propositional calculus. We believe that the other nine algebras also will be of interest from the point of view of Many-Valued Logic, since each of them can provide a new interpretation for the implication connective; for example, it is reasonable to have $F \to T = U$, $F \to U = U$, and $U \to T = U$, where T, F, U stand respectively for "true", "false" and "unsure", and others. We also prove in Section 4 that there are 160 (non-isomorphic) semi-Heyting algebras on a 4-element chain.

We prove in Section 5 that congruences on semi-Heyting algebras are determined by filters—one of the important tools used to investigate these algebras. It follows immediately that the variety $\mathscr{SH}$ of semi-Heyting algebras has equationally definable principal congruences (EDPC) and (hence) Congruence Extension Property (CEP). It is also shown that $\mathscr{SH}$ is arithmetical. Section 6 characterizes the directly indecomposables in $\mathscr{SH}$. Section 7 proves that, up to isomorphism, $\mathbf{2}$ and $\bar{\mathbf{2}}$ are the only simple algebras in $\mathscr{SH}$ which are also primal, and characterizes the subdirectly irreducibles in $\mathscr{SH}$. We introduce in Section 8 several important subvarieties of $\mathscr{SH}$. Section 9 focuses on the subvariety $\mathscr{BSH}$ of Boolean semi-Heyting algebras. It is shown that $\mathscr{BSH}$ is, in fact, the variety generated by $\mathbf{2}$ and $\bar{\mathbf{2}}$. We conclude Section 9 by giving a direct proof that $V(\bar{\mathbf{2}})$ is term-equivalent to the variety of Boolean rings. We should mention here that it is a well known fact that the varieties generated by two primal algebras are categorically equivalent.

In Section 10 we focus on the subvariety $\mathscr{SSH}$ of Stone semi-Heyting algebras. We give a characterization of $\mathscr{SSH}$ and present two properties of subdirectly irreducible algebras in $\mathscr{SSH}$. Using these properties, Section 11 presents small equational bases for each of the ten 3-element semi-Heyting algebras. Our equational basis for the 3-element Heyting algebra seems to be new. In Section 12 we give several new axiom systems for the variety of Heyting algebras by augmenting the axioms for $\mathscr{SH}$ with a single new axiom in each case, as well as some axiom systems not based on the axioms of $\mathscr{SH}$. In Section 13 we define semi-Brouwerian algebras, semi-Heyting semilattices, and semi-Brouwerian semilattices as generalizations of semi-Heyting algebras, and point out which of the results of this paper remain true for each of these classes of algebras. We conclude, in Section 14, with a few open problems, which, we hope, will stimulate further interest in this area.

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2. Semi-Heyting algebras: Arithmetical Properties

We start by recalling some well known definitions and results. For the basic notation and results, we refer the reader to the standard references [1], [3], [4], [5], [11], and [27].

The following definition of Heyting algebras is taken from [1].

Definition 2.1. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a *Heyting algebra* if the following conditions hold:

(H1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with $0, 1$
(H2) $x \wedge (x \to y) \approx x \wedge y$
(H3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$
(H4) $(x \wedge y) \to x \approx 1$.

Heyting algebras, as is well known, are pseudocomplemented (with $x^* = x \to 0$ as the pseudocomplement of x) in the sense of the following definition.

Definition 2.2. An algebra $\mathbf{L} = \langle L, \vee, \wedge, ^*, 0, 1\rangle$ is a *pseudocomplemented lattice (p-lattice or p-algebra)*, where * is unary, if the following hold:

(PS1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with $0, 1$
(PS2) $x \wedge (x \wedge y)^* \approx x \wedge y^*$
(PS3) $0^* \approx 1$ and $1^* \approx 0$.

For more information on p-lattices, see, for example, [1], [9], [11], [27] and [31]. For L a p-lattice, $B(L)$ denotes the set of closed ($a^{**} = a$) elements and $D(L)$ the set of dense ($a^* = 0$) elements. Note that $\langle B(L), \sqcup, \wedge, ^*, 0, 1\rangle$ is a Boolean algebra, where $a \sqcup b = (a^* \wedge b^*)^*$.

The following definition is central to this paper.

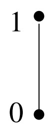
Definition 2.3. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a *semi-Heyting algebra* if the following conditions hold:

(SH1) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with $0, 1$
(SH2) $x \wedge (x \to y) \approx x \wedge y$
(SH3) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$
(SH4) $x \to x \approx 1$.

One can also define the notions of dual semi-Heyting algebras and double semi-Heyting algebras, the latter of which is considered in our paper [34]. The observation that the identity (SH4) is a special case of the identity (H4) shows that Heyting algebras form a subvariety of the variety of semi-Heyting algebras. We denote by $\mathscr{SH}$ the variety of semi-Heyting algebras and by $\mathscr{H}$ the (sub)variety of Heyting algebras.

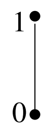
The algebras $\mathbf{2}$ and $\bar{\mathbf{2}}$, which have the two-element chain as their lattice reduct and whose $\to$ operation is defined in Figure 1, are two important examples of semi-Heyting algebras. One easily verifies that $\mathbf{2}$ is a Heyting algebra (which is also a Boolean algebra), while $\bar{\mathbf{2}}$ is not. In the rest of the paper we often write L for $\mathbf{L}$.

$\mathbf{2}$:



$\to$	0	1
0	1	1
1	0	1

$\bar{\mathbf{2}}$:



$\to$	0	1
0	1	0
1	0	1

Figure 1

In the rest of this section, L denotes an arbitrary semi-Heyting algebra. We now present some useful arithmetical properties of semi-Heyting algebras.

Theorem 2.4. *Let $a,b,c,x \in L$. Then*

(a) $1 \to a = a$
(b) $a \leq b$ *implies* $a \wedge (b \to c) = a \wedge c$
(c) $a \leq b$ *implies* $a \leq b \to 1$
(d) $a \leq b$ *implies* $a \leq a \to b$
(e) $b \to c \geq b \wedge c$
(f) $a \leq b$ *implies* $a \leq b \to a$
(g) $a \leq b$ *and* $a \leq c$ *imply* $a \leq b \to c$
(h) $x \leq a \to b$ *implies* $x \wedge a \leq b$
(i) $a \to b = 1$ *implies* $a \leq b$
(j) $a = b$ *iff* $(a \to b) \wedge (b \to a) = 1$
(k) $a = b$ *iff* $(a \vee b) \to (a \wedge b) = 1$
(l) $a \leq (a \wedge b) \to b$
(m) $a \leq a \to 1$
(n) $a \leq b \leq c$ *implies* $b \wedge (a \to c) = b \wedge (a \to b)$
(o) $a \leq (a \to b) \to b$
(p) $a \leq [(a \wedge b) \to b] \to a$
(q) $a \leq a \to (b \to (a \wedge b))$
(r) $a \wedge (0 \to 1) = a \wedge (0 \to a)$
(s) $a \leq 0 \to 1$ *iff* $a \leq 0 \to a$

(t) $b \le a \to 1$ *iff* $b \le (a \wedge b) \to b$
(u) $b \to (a \wedge b) \le (a \wedge b) \to b$
(v) $a \to b \le a \to (a \wedge b)$.

Proof. The conclusions (a) and (d) follow immediately from (SH2), while (b) follows from (SH3) and (SH2), and (c) is a special case of (b). To prove(e), we use (SH3) and (SH4) to get $b \wedge c \wedge (b \to c) = b \wedge c \wedge [(b \wedge c) \to (b \wedge c)] = b \wedge c$, and (f) is a special case of (e). It is clear that (SH3) and (SH4) imply (g). Now, $x \wedge a \wedge b = x \wedge a \wedge (a \to b) = x \wedge a$, as $x \le a \to b$, implying (h), from which (i) and (j) follow immediately, and also (i) implies (k). Next, observe that (l) follows form (SH3) and (SH4), and (m) is a special case of (l). Also, (n) is immediate from (SH3). Observe that $a \wedge ((a \to (a \to b)) \to (a \wedge b)) = a$, using (SH2), (SH3) and (SH4), thus proving (o), and the proof of (p) is similar. To verify (q), we use (SH2), (SH3) and (SH4) to get $a \wedge [a \to (b \to (a \wedge b))] = a \wedge [b \to (a \wedge b)] = a \wedge [(a \wedge b) \to (a \wedge b)] = a$. (r) follows from (n), and (s) is immediate from (r). We note that (t) is immediate from $b \wedge (a \to 1) = b \wedge [(b \wedge a) \to b]$. To prove (u),

$$\begin{aligned}
&[b \to (a \wedge b)] \wedge [(a \wedge b) \to b] \\
&\quad = [b \to (a \wedge b)] \wedge [((a \wedge b) \wedge (b \to (a \wedge b))) \to (b \wedge (b \to (a \wedge b)))] && \text{by (SH3)} \\
&\quad = [b \to (a \wedge b)] \wedge [((b \wedge b) \wedge ((a \wedge b) \to (a \wedge b))) \to (a \wedge b)] \\
&\quad = [b \to (a \wedge b)] \wedge [(a \wedge b) \to (a \wedge b)] \\
&\quad = b \to (a \wedge b).
\end{aligned}$$

Finally, observe that (v) follows from (SH3) and (SH4). □

Remark 2.5. The converse statements of (h) and (i), true in Heyting algebras, do not in general hold in Semi-Heyting algebras—hence the name.

In the rest of this section we give other characterizations of $\mathscr{SH}$. Let L denote the algebra $\langle L, \vee, \wedge, \to, 0, 1\rangle$.

Theorem 2.6. *The following are equivalent:*
(a) *L is a semi-Heyting algebra*
(b) *L satisfies* (SH1), (SH2), (SH3) *and*
(SH5) $x \le (x \to y) \to y$.

Proof. (a) implies (SH5) by Theorem 2.4(o). Suppose (b) holds. First, observe that (SH2) implies $1 \to x \approx x$. Then $1 \le (1 \to x) \to x \approx x \to x$ by (SH5). Thus (SH4) holds, implying (a). □

Theorem 2.7. *The following are equivalent:*
(a) *L is a semi-Heyting algebra*
(b) *L satisfies* (SH1), (SH3), (SH4), and
(SH6) $x \wedge (x \to y) \le x \wedge y$.

Proof. It suffices to prove that (b) implies (SH2). First, we observe that $x \to y \ge x \wedge y$, using (SH3) and (SH4) as in the proof of Theorem 2.4(e). Then

$$\begin{aligned}
x \wedge (x \to y) &\approx x \wedge [x \to (x \wedge y)] && \text{by (SH3)} \\
&\ge x \wedge y.
\end{aligned}$$

Next, use (SH6) to conclude (SH2) holds. □

Theorem 2.8. *If $L \in \mathscr{SH}$ then L satisfies*

(SH7) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to z]$.

Proof. $x \wedge [(x \wedge y) \to z]$

$$\begin{aligned} &\approx x \wedge [(x \wedge y) \to (x \wedge z)] && \text{by (SH3)} \\ &\approx x \wedge (y \to z) && \text{by (SH3)} \end{aligned}$$

So, (SH7) holds. □

Theorem 2.9. *If $L \in \mathscr{SH}$ then L satisfies*

(SH8) $x \wedge (y \to z) \approx x \wedge [y \to (x \wedge z)]$.

Proof. $x \wedge [y \to (x \wedge z)]$

$$\begin{aligned} &\approx x \wedge [(x \wedge y) \to (x \wedge z)] && \text{by (SH3)} \\ &\approx x \wedge (y \to z) && \text{by (SH3),} \end{aligned}$$

proving (SH8). □

It is not hard to find examples (on a 3-element chain, for instance) to show that the converses of the preceding two theorems do not hold.

Corollary 2.10. *Let $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ be an algebra. Then the following are equivalent:*

(a) *$\mathbf{L}$ is a semi-Heyting algebra*
(b) *L satisfies* (SH1), (SH2), (SH4), (SH7) and (SH8).

Proof. (a) implies (b) by theorems 2.8 and 2.9. Suppose (b) holds. Now,

$$\begin{aligned} x \wedge (y \to z) &\approx x \wedge [y \to (x \wedge z)] && \text{by (SH8).} \\ &\approx x \wedge [(x \wedge y) \to (x \wedge z)] && \text{by (SH7).} \end{aligned}$$

Hence (SH3) holds. □

3. The Lattice Structure

In this section we investigate the structure of the lattice reduct of a semi-Heyting algebra.

Theorem 3.1. *Let $L \in \mathscr{SH}$ with $a, b \in L$. For $c \in [a,b]$ we define $c^{*ab} = (c \to a) \wedge b$. Then the algebra $\langle [a,b], \vee, \wedge, ^{*ab}, a, b\rangle$ is a pseudocomplemented lattice.*

Proof. It suffices to verify the conditions (PS2) and (PS3) of Definition 2.2. Let $c, d \in [a,b]$. We first note that $c^{*ab} \geq a$ in view of part (g) of Theorem 2.4, since $a \leq c$, implying that $c^{*ab} \in [a,b]$. Next, note that $a^{*ab} = (a \to a) \wedge b = b$ by (SH4), and also, $b^{*ab} = (b \to a) \wedge b =$

$b \wedge a = a$ by (SH2), proving (PS3) of Definition 2.2. Next,

$$\begin{aligned} c \wedge (c \wedge d^{*ab}) &= c \wedge [((c \wedge d) \to a) \wedge b] \\ &= c \wedge [(c \wedge d) \to a] && \text{since } c \leq b \\ &= c \wedge [(c \wedge d) \to (c \wedge a)] && \text{as } a \leq c \\ &= c \wedge (d \to a) && \text{by (SH3)} \\ &= c \wedge b \wedge (d \to a), && \text{as } c \leq b \\ &= c \wedge d^{*ab}. \end{aligned}$$

Hence (PS2) of Definition 2.2 holds, and the proof is complete. □

Theorem 3.1 has several interesting consequences. For example, we can now derive the property (1) mentioned in the Introduction.

Corollary 3.2. *Let* $\mathbf{L} \in \mathscr{S}\mathscr{H}$. *Then the algebra* $\langle L, \vee, \wedge, ^*, 0, 1\rangle$, *where* $c^* = c \to 0$ *for* $c \in L$, *is a pseudocomplemented lattice.*

Proof. Take $a = 0$ and $b = 1$ in Theorem 3.1. □

Thus the concepts and results of the fairly well developed theory of p-algebras become available for the investigation of semi-Heyting algebras.

Corollary 3.3. *Let* $L \in \mathscr{S}\mathscr{H}$. *Then L is modular iff L is distributive.*

Proof. Observe from Theorem 3.1 that the lattice M_5 cannot be embedded as a sublattice in L. □

Corollary 3.4. *Let* $a, b, c \in L$. *Then*

(i) $a \wedge (a \to b)^* = a \wedge b^*$
(ii) $a^{**} \wedge (a \to b)^{**} = a^{**} \wedge b^{**}$
(iii) *If a is dense, then* $(a \to b)^* = b^*$
(iv) *If a and b are dense then* $a \to b$ *is also dense*
(v) $0 \to 1 = 0$ *iff* $0 \to a \leq a^*$; *hence*
If a is dense, then $0 \to 1 = 0 \quad iff \quad 0 \to a = 0$.
(vi) *If L is chain-based and* $L \models 0 \to 1 \approx 0$ *then* $0 \to a = 0$, *for every* $a \in L - \{0\}$.
(vii) $a^* \leq 0 \to a$. *In particular* $a^* \leq 0 \to a^{**}$
(viii) $a \leq 0 \to a^*$
(ix) $a \to a^* \leq a^* \leq a^{**} \to a$
(x) *If* $0 \to a^* \geq a^*$ *then* $a \to a^* = a^*$
(xi) $a \leq a^{**} \to a$ *and* $a^* \leq a^{**} \to a$
(xii) $a \leq a \to a^{**}$
(xiii) $a^* \leq a \to a^{**}$ *and* $a^{**} \geq a^* \to a$
(xiv) $a \vee a^* \leq a \to a^{**}$; *hence,* $a \to a^{**} \in D(L)$
(xv) $c \leq a$ *implies* $c \wedge (a \to b^*) = c \wedge b^*$
(xvi) $a \leq 0 \to a^{**}$ *iff* $a \leq 0 \to a$
(xvii) $b \wedge (a \to b^*) = b \wedge a^*$
(xviii) *If a is dense, then* $a \to b^* \leq b^*$
(xix) $a \leq b^*$ *implies* $a \to b^* \leq b^*$
(xx) $0 \to a = 0$ *implies* $0 \to 1 \leq a*$

(xxi) *If* $a \wedge b = 0$ *then* $a \to b \leq a^*$.

Proof.

$$
\begin{aligned}
a \wedge (a \to b)^* &= a \wedge [(a \to b) \to 0] \\
&= a \wedge [(a \wedge (a \to b)) \to 0] && \text{by (SH3)} \\
&= a \wedge [(a \wedge b) \to 0] && \text{by (SH2)} \\
&= a \wedge (a \wedge b)^* \\
&= a \wedge b^*,
\end{aligned}
$$

proving (i). For (ii), use (SH2) and $(x \wedge y)^{**} \approx x^{**} \wedge y^{**}$. It is clear that (iii) and (iv) follow from (ii). From $0 \to 1 = 0$ and $a \wedge (0 \to 1) = a \wedge (0 \to a)$ in view of (SH3), we get $a \wedge (0 \to a) = 0$, so that $0 \to a \leq a^*$. The other half of (v) is true by setting $a = 1$, thus (v) holds, from which (vi) is immediate. Observe that $a^* \wedge (0 \to a) = a^* \wedge (0 \to 0) = a^*$, proving (vii); hence we get $a \leq a^{**} \leq 0 \to a^*$, and (viii) follows. Notice that $a \wedge (a \to a^*) = a \wedge a^* = 0$, so $a \to a^* \leq a^*$. Also, $a^* \wedge (a^{**} \to a) = a^*$; hence (ix) holds. For (x) use (ix) and $a^* \wedge (a \to a^*) = a^* \wedge (0 \to a^*)$. Next, $a \wedge (a^{**} \to a) = a$ by (SH3) and (SH4), and $a^* \wedge (a^{**} \to a) = a^* \wedge (0 \to 0) = a^*$, proving (xi), and the proof of (xii) is similar. From $a^* \wedge (a \to a^{**}) = a^* \wedge (0 \to 0) = a^*$, we conclude the first half of (xiii), and for the second half, from $a^* \wedge (a^* \to a) = a^* \wedge a = 0$ we get $a^* \to a \leq a^{**}$. (xiv) is immediate from (xiii). To prove (xv), we note that $c \wedge (a \to b^*) = c \wedge [(c \wedge a) \to (c \wedge b^*)] = c \wedge [c \to (c \wedge b^*)] = c \wedge b^*$. (xvi) follows from $a \wedge (0 \to a^{**}) = a \wedge (0 \to a)$. The proof of (xvii) is similar to that of (i), from which (xviii) follows immediately. If $a^* \leq b^*$ then $b \wedge a^* = 0$, so (xix) follows from (xvii). For (xx) we know $a \wedge (0 \to 1) = a \wedge (0 \to a)$ and $a \wedge (a \to b) = a \wedge b = 0$, so $(a \wedge b) \leq a^*$, thus proving (xxi). □

We now improve Corollary 3.3 and verify property (2) mentioned in the Introduction. We say L is *distributive* if its lattice reduct is distributive.

Theorem 3.5. *Let* $L \in \mathscr{SH}$. *Then* L *is distributive.*

Proof. In view of Corollary 3.3, it suffices to show that N_5 is not a sublattice of (the lattice reduct of) L. Suppose N_5 is a sublattice of L (see Figure 2).

$\mathbf{N}_5$:

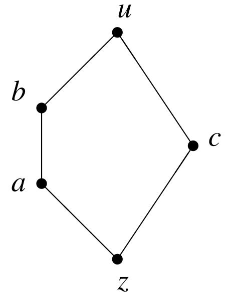


Figure 2

Then

$$
\begin{aligned}
c \wedge (b \to a) &= c \wedge [(c \wedge b) \to (c \wedge a)] && \text{by (SH3)} \\
&= c \wedge [(c \wedge a) \to (c \wedge a)] && \text{since } \mathrm{c} \wedge \mathrm{b} = c \wedge a \\
&= c && \text{by (SH4).}
\end{aligned}
$$

Thus $c \leq b \to a$. Also, we know $a \leq (b \to a)$ by Theorem 2.4(f), as $a \leq b$. Hence, $a \vee c \leq b \to a$. But then

$$
\begin{aligned}
a = b \wedge a = b \wedge (b \to a) & && \text{by (SH2)} \\
\geq b \wedge (a \vee c) = b. &
\end{aligned}
$$

Thus $a \geq b$, which is a contradiction (see Figure 2), completing the proof. □

The following theorem generalizes another well known property of Heyting algebras.

Theorem 3.6. *Let $L \in \mathcal{SH}$ and let $a,b \in L$ with $a \leq b$. For $c,d \in [a,b]$, define $c \to^{ab} d = (c \to d) \wedge b$. Then the algebra $\mathbf{L_0} = \langle [a,b], \vee, \wedge, \to^{ab}, a, b \rangle$ is a semi-Heyting algebra. Furthermore, if L is a Heyting algebra, then $\mathbf{L_0}$ is also a Heyting algebra.*

Proof. Let $c,d,e \in [a,b]$. First, we note that $c \to^{ab} d \in [a,b]$ by Theorem 2.4(g). Now,

$$
\begin{aligned}
c \wedge (c \to^{ab} d) &= c \wedge (c \to d) \wedge b \\
&= c \wedge d \wedge b \\
&= c \wedge d,
\end{aligned}
$$

so (SH2) holds in $\mathbf{L_0}$. Next,

$$
\begin{aligned}
e \wedge (c \to^{ab} d) &= e \wedge (c \to d) \wedge b \\
&= e \wedge [(e \wedge c) \to (e \wedge d)] \wedge b && \text{by (SH3)} \\
&= e \wedge [(e \wedge c) \to^{ab} (e \wedge d)],
\end{aligned}
$$

so (SH3) holds in $\mathbf{L_0}$. Finally, $c \to^{ab} c = (c \to c) \wedge b = b$, thus (SH4) holds in $\mathbf{L_0}$. Hence $\mathbf{L_0}$ is a semi-Heyting algebra. For $L \in \mathcal{H}$, $(c \wedge d) \to^{ab} d = [(c \wedge d) \to d] \wedge b = b$. Thus $\mathbf{L_0}$ is also a Heyting algebra. Hence the proof is complete. □

Recall that $a \to b = a^* \vee b$ in a Boolean algebra. On the other hand, in a semi-Heyting algebra, we have the following

Theorem 3.7. *Let $L \in \mathcal{SH}$ and $a,b \in L$. Then $(a \to b)^{**} \leq a^* \sqcup b^{**}$.*

Proof.

$$
\begin{aligned}
(a \to b)^{**} &= (a^{**} \sqcup a^*) \wedge (a \to b)^{**} \\
&= [a^{**} \wedge (a \to b)^{**}] \sqcup [a^* \wedge (a \to b)^{**}] \\
&= [a \wedge (a \to b)]^{**} \sqcup [a^* \wedge (a \to b)^{**}] \\
&= (a \wedge b)^{**} \sqcup [a^* \wedge (a \to b)^{**}] \\
&= [(a \wedge b)^{**} \sqcup a^*] \wedge [(a \wedge b)^{**} \sqcup (a \to b)^{**}] \\
&= [(a^{**} \wedge b^{**}) \sqcup a^*] \wedge [(a^{**} \wedge b^{**}) \sqcup (a \to b)^{**}] \\
&= (b^{**} \sqcup a^*) \wedge [(a^{**} \sqcup (a \to b)^{**}) \wedge (b^{**} \sqcup (a \to b)^{**})] \\
&= (b^{**} \sqcup a^*) \wedge (a^* \wedge (a \to b)^*)^* \wedge (b^* \wedge (a \to b)^*)^*,
\end{aligned}
$$

proving the theorem. $\square$

It should be noted that the above proof proves a much stronger statement than the one given in the previous theorem.

Theorem 3.8. *Let $L \in \mathscr{SH}$ and $a, b \in L$. Then*

$$(a \vee a^*) \wedge (a \to b) \leq a^* \vee b.$$

Proof. $(a \vee a^*) \wedge (a \to b)$

$$\begin{aligned}
&= (a \wedge (a \to b) \vee [a^* \wedge (a \to b)] && \text{by distributivity}\\
&= (a \wedge b) \vee [a^* \wedge (a \to b)] && \text{by (SH2)}\\
&= [(a \wedge b) \vee a^*] \wedge [(a \wedge b) \vee (a \to b)] && \\
&= (a \vee a^*) \wedge (a^* \vee b) \wedge (a \to b) && \text{by Theorem 2.4(e).}
\end{aligned}$$

$\square$

Theorem 3.9. *Let $L \in \mathscr{SH}$ and $a, b, c \in L$. Then*

(1) $a^{**} \wedge (a \to b)^* = a^{**} \wedge b^*$
(2) $a^{**} \wedge (b \to c)^* = a^{**} \wedge [(a \wedge b) \to (a \wedge c)]^*$
(3) $b^* \wedge (a \to b) = b^* \wedge a^*$.

Proof. From (SH2) we get $a^* \sqcup (a \to b)^* = a^* \sqcup b^*$, hence it follows, using distributivity of $B(L)$ that $a^{**} \wedge (a \to b)^* = a^{**} \wedge b^*$, proving (1). Use (SH3) to prove (2). For (3),

$$\begin{aligned}
b^* \wedge (a \to b) &= b^* \wedge (b^* \wedge a)^* && \text{by (SH3)}\\
&= b^* \wedge a^* && \text{by (PS2).}
\end{aligned}$$

$\square$

4. Chain-based semi-Heyting algebras

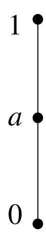
We say $\mathbf{L} \in \mathscr{SH}$ is a *semi-Heyting chain* if the lattice reduct of $\mathbf{L}$ is a chain. The following theorem and Theorem 4.4 show that there is an abundant supply of semi-Heyting chains that are not Heyting chains.

Theorem 4.1. *There are, up to isomorphism, exactly* 10 3*-element semi-Heyting chains, as defined in figure* 3.

$\mathbf{L}_1$:

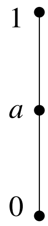
$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\mathbf{L}_2$:



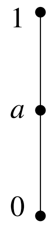
$\to$	0	a	1
0	1	a	1
a	0	1	1
1	0	a	1

$\mathbf{L}_3$:



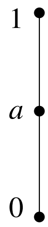
$\to$	0	a	1
0	1	1	1
a	0	1	a
1	0	a	1

$\mathbf{L}_4$:



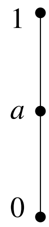
$\to$	0	a	1
0	1	a	1
a	0	1	a
1	0	a	1

$\mathbf{L}_5$:



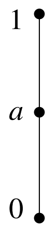
$\to$	0	a	1
0	1	a	a
a	0	1	1
1	0	a	1

$\mathbf{L}_6$:



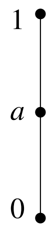
$\to$	0	a	1
0	1	1	a
a	0	1	1
1	0	a	1

$\mathbf{L}_7$:



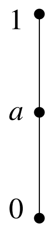
$\to$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

$\mathbf{L}_8$:



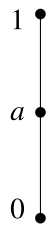
$\to$	0	a	1
0	1	1	a
a	0	1	a
1	0	a	1

$\mathbf{L}_9$:



$\to$	0	a	1
0	1	0	0
a	0	1	1
1	0	a	1

$\mathbf{L}_{10}$:



$\to$	0	a	1
0	1	0	0
a	0	1	a
1	0	a	1

Figure 3

Proof. It is routine to verify that $\mathbf{L}_i \in \mathscr{SH}$ for $i = 1, 2, \ldots, 10$. Now, let $\mathbf{L}$ be a three element chain $\{0, a, 1\}$ with $0 < a < 1$. In view of (SH4), Theorem 2.4(a) and Corollary 3.2, the operation $\to$ is uniquely determined except for $0 \to a$, $0 \to 1$ and $a \to 1$. Since

$a \leq a \to 1$ by Theorem 2.4(m), we get $a \to 1 = a, 1$. Let $0 \to 1 = 0$. Then $0 \to a = 0$ by Theorem 2.4(r), since 0 is $\wedge$-irreducible. Thus $\mathbf{L} \cong \mathbf{L}_9$ or $\mathbf{L} \cong \mathbf{L}_{10}$. Next, let $0 \to 1 \geq a$. Then $0 \to a \geq a$ by Theorem 2.4(r). Also, we already know that $a \to 1 = a, 1$. Thus $\mathbf{L} \cong \mathbf{L}_i$ for $i = 1, 2, \ldots 8$. □

In Section 11, we give equational bases for these algebras. We note that $\mathbf{L}_1$ is a Heyting algebra and the rest are not.

Lemma 4.2. *Let $x, y, z \in L \in \mathscr{SH}$ such that $x < y \leq z$ and x is $\wedge$-irreducible. Then*

(i) $x \to z = x$ *iff* $x \to y = x$.
(ii) $0 \to 1 = x$ *iff* $0 \to y = x$.
(iii) $y \to x = x$.

Proof. Let $x \to z = x$. Since $x < y$, it follows that $x = y \wedge x = y \wedge (x \to z) = y \wedge [(x \wedge y) \to (y \wedge z)] = y \wedge (x \to y)$, implying $x \to y = x$ since x is $\wedge$-irreducible. Conversely, suppose $x \to y = x$. Then $y \wedge (x \to z) = y \wedge (x \to y) = y \wedge x = x$, hence $x \to z = x$, proving (i). (ii) follows from $y \wedge (0 \to 1) = y \wedge (0 \to y)$, as x is $\wedge$-irreducible. Finally, from $y \wedge (y \to x) = y \wedge x = x$ it follows that and $y \to x = x$, since x is $\wedge$-irreducible, proving (iii). □

The following theorem is immediate from part (iii) of the above lemma.

Theorem 4.3. *Let $\mathbf{L} \in \mathscr{SH}$ whose lattice-reduct is the n-element chain: $0 < a_1 < a_2 < \ldots < a_{n-2} < 1$. Then the lower half, including the main diagonal, of the operation table of $\to$ of $\mathbf{L}$ is uniquely determined as in* Figure 4.

$\to$	0	a_1	a_2	$\cdots$	a_{n-3}	a_{n-2}	1
0	1						
a_1	0	1					
a_2	0	a_1	1				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
a_{n-3}	0	a_1	a_2	$\cdots$	1		
a_{n-2}	0	a_1	a_2	$\cdots$	a_{n-3}	1	
1	0	a_1	a_2	$\cdots$	a_{n-3}	a_{n-2}	1

Figure 4

Theorem 4.4. *There are, up to isomorphism,* 160 *semi-Heyting algebras on a* 4*-element chain.*

Proof. Let $\mathbf{L} \in \mathscr{SH}$ with $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. In view of Theorem 4.3, we need only determine the upper half of the operation table of $\to$. From Theorem 2.4(m) we have $b \to 1 \geq b$, $a \to 1 \geq a$. Now if $a \to 1 = a$ then $a \to b = a$ by Lemma 4.2(i). Also, if $a \to 1 \in \{b, 1\}$ then $a \to b \in \{b, 1\}$. Thus we are led to the tree given in Figure 5, each of whose paths describes the partially complete tables of several possible $\to$ operations. The following notation is used in Figures 5 and 6 for convenience:

α_1 denotes: $[b \to 1 = b, 1]$
α_2 denotes: $[a \to 1 = b, 1]$ & $[a \to b = b, 1]$
α_3 denotes: $[a \to 1 = a]$ & $[a \to b = a]$
α_4 denotes: $[0 \to 1 = b, 1]$ & $[0 \to b = b, 1]$ & $[0 \to a = a, b, 1]$
α_5 denotes: $[0 \to 1 = a]$ & $[0 \to b = a]$ & $[0 \to a = a, b, 1]$
α_6 denotes: $[0 \to 1 = 0]$ & $[0 \to b = 0]$ & $[0 \to a = 0]$

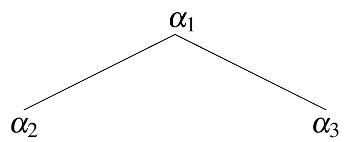


Figure 5

Since $0 \to 1 \in \{0, a, b, 1\}$, we shall now consider the following cases:

Let $0 \to 1 = b, 1$. Then $0 \to b = b, 1$ and $0 \to a = a, b, 1$. Next, let $0 \to 1 = a$. Then $0 \to b = a$ and $0 \to a = a, b, 1$. Finally, let $0 \to 1 = 0$. This would imply $0 \to b = 0$ and $0 \to a = 0$. Thus we are led to the following full tree which extends the earlier partial tree with the property that its paths determine the upper half of all possible operation tables for $\to$, and hence all tables for $\to$.

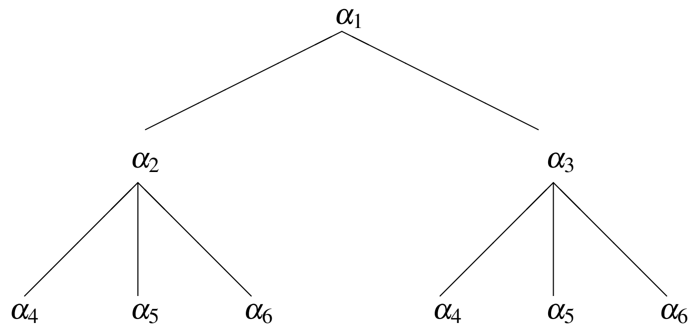


Figure 6

Thus the number of non-isomorphic 4-element semi-Heyting chains is

$$\begin{aligned}
&= 2\cdot(2^2)\cdot[2^2\cdot 3+1^2\cdot 3+1^3]+2\cdot(1^2)\cdot[2^2\cdot 3+1^2\cdot 3+1^3]\\
&= 2^3\cdot[16]+2\cdot[16]\\
&= 16\cdot[8+2]\\
&= 160.
\end{aligned}$$

This proves the theorem. □

Corollary 4.5. *Let $\mathscr{V}$ be the subvariety of $\mathscr{SH}$ defined by: $x \to 1 \approx 1$. Then there are* 12 *algebras in $\mathscr{V}$ with the* 4*-element chain as the lattice reduct.*

Proof. First observe, in view of Lemma 4.2(i), that α_3 (defined in the above proof) is false in a 4-element semi-Heyting chain of $\mathscr{V}$. Hence we need only consider the following subtree of the one in Theorem 4.4.

$$b \to 1 = 1$$

$$\begin{aligned} a &\to 1 = 1\\ a &\to b = b, 1 \end{aligned}$$

$$\begin{aligned} 0 &\to 1 = 1\\ 0 &\to b = b, 1\\ 0 &\to a = a, b, 1 \end{aligned}$$

Figure 7

Thus we have $1\cdot(1\cdot 2)\cdot(1\cdot 2\cdot 3)=12$ different paths. Hence the corollary follows. $\square$

Corollary 4.6. *Let $\mathscr{V}$ be the subvariety of $\mathscr{SH}$ defined by the identity: $y \leq x \to y$. Then there are* 12 *non-isomorphic algebras in $\mathscr{V}$ based on a* 4*-element chain.*

Proof. First note that $\mathscr{V} \models x \to 1 \approx 1$. Then it is not hard to see that one is led to consider the same tree as the one given in the previous corollary. Thus there are 12 non-isomorphic 4-element semi-Heyting chains in $\mathscr{V}$. $\square$

The proofs of the preceding two corollaries show that if $\mathbf{L}$ is a 4-element semi-Heyting chain, then $\mathbf{L} \models y \leq x \to y$ iff $\mathbf{L} \models x \to 1 \approx 1$. We generalize this observation in Section 8 (see Theorem 8.7).

Corollary 4.7. *There is, up to isomorphism, exactly one semi-Heyting algebra on a* 4*-element chain satisfying the identity $x \to 1 \approx x$.*

Proof. We need only consider the following subtree of the tree of Theorem 4.4:

$$b \to 1 = b$$

$$\Big|$$

$$\begin{gathered} a \to 1 = a \\ a \to b = a \end{gathered}$$

$$\Big|$$

$$\begin{gathered} 0 \to 1 = 0 \\ 0 \to b = 0 \\ 0 \to a = 0 \end{gathered}$$

Figure 8

Hence there is, essentially, only one semi-Heyting algebra on a 4-chain that satisfies: $x \to 1 \approx x$. $\square$

Corollary 4.8. *The 4-element semi-Heyting chain satisfying the identity: $x \to 1 \approx x$ also satisfies the commutative identity: $x \to y \approx y \to x$.*

Proof. Use Figure 8 and construct the $\to$ operation table for this algebra. $\square$

5. Congruences and filters

In this section we prove that the congruences on semi-Heyting algebras are determined by filters —an important tool in the investigation of the variety $\mathscr{SH}$.

In the rest of this section L denotes a semi-Heyting algebra, and $F(L)$ denotes the lattice of filters of L.

Definition 5.1. Let $F \in F(L)$. Define a binary relation $\Theta(F)$ on L by
$\langle x,y\rangle \in \Theta(F)$ iff $x \wedge f = y \wedge f$, for some $f \in F$.

Lemma 5.2. $\Theta(F) \in \mathrm{Con(L)}$ *and* $1/\Theta(\mathrm{F}) = \mathrm{F}$.

Proof. Since L is distributive, clearly $\Theta(F)$ is a lattice congruence. Let $\langle a,b\rangle \in \Theta(F)$ and $\langle c,d\rangle \in \Theta(F)$. Then there are f_1, $f_2 \in F$ such that $a \wedge f_1 = b \wedge f_1$ and $c \wedge f_2 = b \wedge f_2$. Hence it follows from (SH3) that

$$\begin{aligned} f_1 \wedge f_2 \wedge (a \to c) &= f_1 \wedge f_2 \wedge [(a \wedge f_1 \wedge f_2) \to (c \wedge f_1 \wedge f_2)] \\ &= f_1 \wedge f_2 \wedge [(b \wedge f_1 \wedge f_2) \to (d \wedge f_1 \wedge f_2)] \\ &= f_1 \wedge f_2 \wedge (b \to d). \end{aligned}$$

Since $f_1 \wedge f_2 \in F$, it follows that $\Theta(F)$ is compatible with $\to$, so that $\Theta(F) \in \mathrm{Con(L)}$. The other half is left to the reader. □

Lemma 5.3. *If $F \in F(L)$ and $a,b \in L$, then*
$\langle a,b\rangle \in \Theta(F)$ *iff* $(a \to b) \wedge (b \to a) \in F$.

Proof. Let $\langle a,b\rangle \in \Theta(F)$, then $\langle a \to b, b \to b\rangle \in \Theta(F)$ and $\langle b \to a, a \to a\rangle \in \Theta(F)$. So $\langle a \to b, 1\rangle \in \Theta(F)$ and $\langle b \to a, 1\rangle \in \Theta(F)$ by (SH4), and hence $\langle (a \to b) \wedge (b \to a), 1\rangle \in \Theta(F)$. Thus $(a \to b) \wedge (b \to a) \in F$. Next, suppose $(a \to b) \wedge (b \to a) \in F$. Then $a \to b \in F$ and $b \to a \in F$. Hence $\langle a \to b, 1\rangle \in \Theta(F)$, which implies $\langle a \wedge (a \to b), a\rangle \in \Theta(F)$, so that $\langle a \wedge b, a\rangle \in \Theta(F)$ by (SH2). Similarly, $\langle a \wedge b, b\rangle \in \Theta(F)$, from which it follows that $\langle a,b\rangle \in \Theta(F)$. □

We are ready to prove property (3) mentioned in the Introduction.

Theorem 5.4. $\mathrm{Con(L)} \cong \mathrm{F(L)}$.

Proof. Observe that the function $\Theta : \mathrm{F(L)} \to \mathrm{Con(L)}$ given in the Definition 5.1 is an isomorphism in view of Lemmas 5.2 and 5.3. □

Remark 5.5. We note that the Theorem 5.4 extends a well known result on Heyting algebras to $\mathscr{SH}$. Our proof also applies to Brouwerian algebras and provides perhaps a less computational proof of the corresponding result for implicative semilattices proved in [23].

Remark 5.6. Since semi-Heyting algebras have lattices as reducts, it follows that the variety $\mathscr{SH}$ is congruence-distributive. In fact, we have the following stronger result.

Corollary 5.7. *The variety $\mathscr{SH}$ has EDPC and hence CEP.*

Here is another improvement of the statement made in Remark 5.6.

Theorem 5.8. *The variety $\mathscr{SH}$ is arithmetical.*

Proof. Consider the ternary term
$p(x,y,z) = [(x \to y) \to z] \wedge [(z \to y) \to x] \wedge (x \vee z)$.
Now,

$$\begin{aligned} p(x,y,y) &= [(x \to y) \to y] \wedge x, && \text{since } y \to y = 1 \text{ and } 1 \to x = x \\ &= x, && \text{since } x \leq (x \to y) \to y. \end{aligned}$$

Also,

$$\begin{aligned} p(x,x,y) &= y \wedge [(y \to x) \to x], && \text{since } x \to x = 1 \text{ and } 1 \to x = x \\ &= y, && \text{since } y \leq (y \to x) \to x. \end{aligned}$$

Thus, $p(x,y,z)$ is a Mal'cev term, so the variety $\mathscr{SH}$ is congruence-permutable, which, together with the Remark 5.6, implies that $\mathscr{SH}$ is arithmetical. □

A careful examination of the preceding proof actually leads us to the following more general result.

Theorem 5.9. *Let $\mathscr{V}$ be a variety of algebras $\mathbf{A} = \langle A, \vee, \wedge, \to, 1\rangle$, where $\to$ is binary, such that the following conditions hold:*

(a) $\langle A, \vee, \wedge, 1\rangle$ *is a lattice with* 1
(b) $1 \to x \approx x$
(c) $x \leq (x \to y) \to y$.

Then $\mathscr{V}$ is arithmetical.

Proof. The proof of the preceding theorem depends only on (b), (c) and the identity $y \to y \approx 1$. Observe that (b) and (c) actually imply $y \to y \approx 1$. Hence it follows from the proof of Theorem 5.8 that $p(x,y,z)$ is a Mal'cev term, implying that $\mathscr{V}$ is congruence permutable. Since $\mathscr{V}$ is congruence-distributive in view of (a) and Remark 5.6, the proof is complete. □

The above theorem would, still, hold if $\vee, \wedge, \to$ and 1 were term functions, instead of fundamental operations. Also, the lattice reducts of algebras in the variety described in Theorem 5.9 need not even be modular. For example, consider the variety generated by the algebra $\langle N_5, \vee, \wedge, \to, 0, 1\rangle$, where N_5 is the 5-element non-modular lattice (see Figure 2) as the lattice-reduct (with z and u renamed as 0 and 1 respectively), and $\to$ is defined by the following table:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	c	1
b	c	a	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Figure 9

6. Directly Indecomposable Semi-Heyting Algebras

In the present section we characterize the directly indecomposable semi-Heyting algebras. Throughout this section $L \in \mathscr{SH}$. We denote the center of L by Cen(L). Recall that Cen(L) is the sublattice of complemented elements of (the lattice reduct of) L.

Definition 6.1. For $a \in L$ we let $\mathbf{L}_a = \langle [0,a], \vee, \wedge, \to^a, 0, a \rangle$, where $c \to^a d = (c \to d) \wedge a$, for $c, d \in [0,a]$. L_a is called the *relativized algebra* of L with respect to a. We will simply write L_a for $\mathbf{L}$.

Lemma 6.2. *Let $a \in L$. Then*

(i) *L_a is a semi-Heyting algebra*
(ii) *The function $f_a : L \to L_a$, defined by $f_a(x) = x \wedge a$, is a homomorphism onto L_a.*

Proof. Observe that (i) is a special case of Theorem 3.6. To prove (ii), use the distributivity and (SH3). □

We now give a characterization of directly indecomposable semi-Heyting algebras.

Lemma 6.3. *If L is directly indecomposable then* $\mathrm{Cen(L)} = \{0,1\}$.

Proof. Suppose $\mathrm{Cen(L)} \neq \{0,1\}$. Let $a \in \mathrm{Cen(L)} - \{0,1\}$. Define $h : L \to L_a \times L_{a^*}$ by $h(x) = \langle f_a(x), f_{a^*}(x) \rangle$. Using Lemma 6.2 it is easy to verify that h is an isomorphism. □

Lemma 6.4. *Let L be a semi-Heyting algebra. If* $\mathrm{Cen(L)} = \{0,1\}$*, then L is directly indecomposable.*

Proof. Suppose L is not directly indecomposable. Then there exists a pair of factor congruences θ and θ_1 different from Δ and ∇ (see [5]). So, $\theta \cap \theta_1 = \Delta$ and $\theta \circ \theta_1 = \nabla$. Then there exists an $a \in L - \{0,1\}$ such that $0 \ \theta \ a \ \theta_1 \ 1$. Hence $\langle a^*, 1 \rangle \in \theta$. Thus $\langle a, 1 \rangle \in \theta_1$ and $\langle a^*, 1 \rangle \in \theta$, so $\langle a \vee a^*, 1 \rangle \in \theta \cap \theta_1 = \Delta$. It follows that $a \vee a^* = 1$. The distributivity of L leads us to conclude that $a^{**} = a$. Thus $a \in \mathrm{Cen(L)}$, showing that $\mathrm{Cen(L)} \neq \{0,1\}$. Hence the lemma is proved. □

Theorem 6.5. *The following are equivalent:*

(i) *L is directly indecomposable*
(ii) $\mathrm{Cen(L)} = \{0,1\}$
(iii) *If $a \in L - \{0,1\}$ then $a \vee a^* < 1$.*

Proof. (i) implies (ii) by Lemma 6.3 and (ii) implies (i) by the preceding lemma. Finally, it is straightforward to verify the equivalence of (ii) and (iii). □

Corollary 6.6. *The class of directly indecomposable members of $\mathscr{SH}$ forms a strictly elementary class.*

Applications of the Theorem 6.5 will be given later in Sections 7, 9 and 10 (see Theorems 7.1, 9.4 and 10.2).

7. Simplicity, Primality and Subdirect Irreducibility in $\mathscr{SH}$

In this section we characterize simples and subdirectly irreducibles in $\mathscr{SH}$. We also prove that $\mathbf{2}$ and $\overline{\mathbf{2}}$ are the only two primal algebras in $\mathscr{SH}$.

The following theorem is an immediate consequence of Theorem 6.5 (or Theorem 5.4).

Theorem 7.1. *The algebras* $\mathbf{2}$ *and* $\overline{\mathbf{2}}$ *are the only simple algebras in the variety* $\mathscr{SH}$.

Corollary 7.2. *The lattice of subvarieties of the variety* $\mathscr{SH}$ *has exactly two atoms, namely* $V(\mathbf{2})$ *and* $V(\overline{\mathbf{2}})$.

We know from Theorem 5.8 that $V(\{\mathbf{2},\overline{\mathbf{2}}\})$ is an arithmetical variety. The following theorem strengthens this.

Theorem 7.3. *The variety* $V(\{\mathbf{2},\overline{\mathbf{2}}\})$ *is a discriminator variety. Hence every non-trivial member of this variety is a Boolean product of* $\mathbf{2}$ *and* $\overline{\mathbf{2}}$.

Proof. It is straightforward to verify that the following term is a discriminator term on $\{\mathbf{2},\overline{\mathbf{2}}\}$:

$$t(x,y,z) = [(x\wedge z)\vee y^{*}]\wedge(x\vee z).$$

□

It is well known that $\mathbf{2}$ is primal. Now we also have the following

Corollary 7.4. $\mathbf{2}$ *and* $\overline{\mathbf{2}}$ *are* (*the only*) *primal algebras in* $\mathscr{SH}$.

As an important application of Theorem 5.4, we characterize the subdirectly irreducible algebras in $\mathscr{SH}$.

Theorem 7.5. *Let* $\mathbf{L}\in\mathscr{SH}$ *with* $|L|\geq 2$. *Then the following are equivalent:*

(1) $\mathbf{L}$ *is subdirectly irreducible.*
(2) $\mathbf{L}$ *has a unique coatom.*

Proof. Suppose (1) holds and let θ be the smallest non-trivial congruence on $\mathbf{L}$. Then, using Theorem 5.4, we let F be the filter corresponding to θ. Then it is easy to see that $F=[a,1)$, where a is a coatom. Since θ is the smallest non-trivial congruence, it can be easily seen, using the isomorphism, that a is the only coatom, proving (2). Since the converse is easily verified, the proof is complete. □

Corollary 7.6. *The class of subdirectly irreducible algebras in* $\mathscr{SH}$ *is strictly elementary.*

8. Some Subvarieties of $\mathscr{SH}$

In this section we initiate the investigation into the structure of the lattice of subvarieties of $\mathscr{SH}$ (the study of which is explored further in the forthcoming paper [33]) by singling out some important subvarieties of $\mathscr{SH}$. We also present several characterizations of some of these varieties in this and later sections.

Definition 8.1.

Subvariety	**Defining base modulo $\mathscr{SH}$**
$\mathscr{FTT}$ (From: "**F**alse implies **T**rue" is **T**rue)	$0 \to 1 \approx 1$
$\mathscr{FTD}$ (From: "**F**alse implies **T**rue" is **D**ense)	$(0 \to 1)^* \approx 0$
$\mathscr{QH}$ (Quasi-Heyting Algebras)	$y \leq x \to y$
$\mathscr{H}$ (Heyting Algebras)	$(x \wedge y) \to x \approx 1$
$\mathscr{SSH}$ (Stone Semi-Heyting Algebras)	$x^* \vee x^{**} \approx 1$
$\mathscr{BSH}$ (Boolean Semi-Heyting Algebras)	$x^{**} \approx x$, or $x \vee x^* \approx 1$
$\mathscr{FTF}$ (From: "**F**alse implies **T**rue" is **F**alse)	$0 \to 1 \approx 0$
$\mathscr{PTP}$ (From: "**P**ossible implies **T**rue" is **P**ossible)	$x \to 1 \approx x$
com$\mathscr{SH}$ (Commutative Semi-Heyting Algebras)	$x \to y \approx y \to x$
$\mathscr{CSH}$ (The subvariety of $\mathscr{S}H$ generated by chains)	? (open)
$\mathscr{CFTT}$ (=$\mathscr{F}TT \cap \mathscr{C}SH$)	? (open)
$\mathscr{CQH}$ (= $\mathscr{QH} \cap \mathscr{CSH}$)	? (open)
$\mathscr{CFTF}$ (=$\mathscr{F}TF \cap \mathscr{C}SH$)	? (open)
$\mathscr{C}$com$\mathscr{SH}$ (=com$SH \cap \mathscr{C}SH$)	? (open)

It follows from Theorem 4.3 (or Corollary 4.8) that there is exactly one commutative semi-Heyting algebra, up to isomorphism, on an n-element chain.

Another basis for $\mathscr{FTT}$ is given below.

Theorem 8.2. *Let* $\mathbf{L} \in \mathscr{SH}$. *Then* $\mathbf{L} \in \mathscr{FTT}$ *iff* $\mathbf{L} \models x \leq 0 \to x$.

Proof. $0 \to 1 \approx 1$ implies $x \wedge (0 \to 1) \approx x$, so $x \wedge (0 \to x) \approx x$ by (SH3). For the converse, take $x = 1$. □

Theorem 8.3. *Let* $\mathbf{L} \in \mathscr{FTF}$ *and* $a, b \in L$. *Then*

(i) $a \wedge b = 0$ *implies* $a \to b = a^* \wedge b^*$
(ii) $0 \to b = b^*$
(iii) $a \to a^* = 0$
(iv) $a^* \to a = 0$.

Proof. From $b \wedge (a \to b) = b \wedge (0 \to b) = b \wedge (0 \to 1) = 0$ we get $a \to b \leq b^*$ in view of Corollary 3.2. Also, $a \to b \leq a^*$ by Corollary 3.4(xxi), thus $a \to b \leq a^* \wedge b^*$. Now we know $b^* \wedge (a \to b) = b^* \wedge a^*$ by Theorem 3.9(c). Hence $a \to b \geq a^* \wedge b^*$, thus proving (i). The rest of the parts follow immediately from (i). □

We now give several characterizations of the variety $\mathscr{FTF}$.

Theorem 8.4. *Let* $\mathbf{L} \in \mathscr{SH}$. *Then the following are equivalent:*

(1) $\mathbf{L} \models 0 \to 1 \approx 0$
(2) $\mathbf{L} \models 0 \to x \leq x^*$
(3) $\mathbf{L} \models 0 \to x \approx x^*$
(4) $\mathbf{L} \models x \wedge (x^* \to y) \approx x \wedge y^*$
(5) $\mathbf{L} \models x^* \to 1 \approx x^*$
(6) $\mathbf{L} \models x^* \to x \approx 0$.

Proof. (1) $\Rightarrow$ (2) by Theorem 8.3. Since we know $x^* \leq 0 \to x$ by Corollary 3.4(vii), we get (2) $\Rightarrow$ (3). Suppose (3) holds, and let $a, b \in L$. Now, we have $a \wedge (a^* \to b) = a \wedge (0 \to (a \wedge b)) = a \wedge (a \wedge b)^* = a \wedge b^*$ using (3), so (3) $\Rightarrow$ (4). Next, $x = 1$ and $y = 1$ yields (4) $\Rightarrow$ (1). Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Using (3) we get $x \wedge (x^* \to 1) = x \wedge (0 \to x) = x \wedge x^* = 0$, so $x^* \to 1 \leq x^*$. The reverse inequality holds by Theorem 2.4(m), proving (5). (5) $\Rightarrow$ (1) is trivial. Hence (3) $\Rightarrow$ (5) $\Rightarrow$ (1) $\Rightarrow$ (3). Suppose (3) holds. Then $x \wedge (x^* \to x) = x \wedge (0 \to x) = x \wedge x^* = 0$ by (3), hence $x^* \to x \leq x^*$. Also, we know $x^* \to x \leq x^{**}$ by Corollary 3.4(xiii), so $x^* \to x \leq x^* \wedge x^{**} = 0$. Thus it follows that (3) $\Rightarrow$ (6) $\Rightarrow$ (1) $\Rightarrow$ (3), which completes the proof. □

Let $\mathscr{CPTP}$ denote the subvariety of $\mathscr{PTP}$ generated by semi-Heyting chains. We will now improve Corollary 4.8. First we need to prove the following.

Lemma 8.5. *Let $L \in \mathscr{CPTP}$ and let $a, b \in L$ with $a < b$. Then $a \to b = a$.*

Proof. From $a \to 1 = a$ and (SH3) we have $b \wedge (a \to b) = b \wedge (a \to 1) = b \wedge a = a$, from which it follows that $a \to b = a$, since a is $\wedge$-irreducible. □

Theorem 8.6. $\mathscr{CPTP}$ = $\mathscr{C}$com$\mathscr{SH}$.

Proof. Let $L \in \mathscr{CPTP}$ and let $a, b \in L$. Let $L \in \mathscr{CPTP}$ and let $a, b \in L$. If $a = b$ then clearly $a \to b = 1 = b \to a$ by (SH4). So, we assume $a \neq b$. Since L is a chain, we may further assume that $a < b$. Then by the preceding lemma, we have $a \to b = a$. Also, we already know $b \to a = a$ by Lemma 4.2(iii). Thus $a \to b = b \to a$. □

We conclude this section by giving another basis for $\mathscr{QH}$.

Theorem 8.7. *Another equational basis for $\mathscr{QH}$, modulo $\mathscr{SH}$, is given by the identity: $x \to 1 \approx 1$.*

Proof. Let $\mathscr{V}$ be the subvariety of $\mathscr{SH}$ defined by the identity: $x \to 1 \approx 1$. It is clear that $\mathscr{QH} \models x \to 1 \approx 1$. Now, let $L \in \mathscr{V}$. For $a, b \in L$ we have $b = b \wedge 1 = b \wedge (a \to 1) = b \wedge ((a \wedge b) \to b) = b \wedge (a \to b)$, implying $b \leq a \to b$. Thus $\mathscr{V} = \mathscr{QH}$. □

9. Boolean Semi-Heyting Algebras

In this section we show that $V(\mathbf{2},\ \bar{\mathbf{2}})$ is the variety $\mathscr{BSH}$ of Boolean semi-Heyting algebras. We also give several equational bases for this variety and for its subvarieties $V(\mathbf{2})$ and $V(\bar{\mathbf{2}})$.

Recall that $\mathbf{L} \in \mathscr{SH}$ is Boolean iff $\mathbf{L} \models x^{**} \approx x$. The algebras $\mathbf{2}$ and $\bar{\mathbf{2}}$ are Boolean. If $\mathbf{L}$ is Boolean, then it is clear that Cen $(\mathbf{L}) = B(\mathbf{L}) = \mathbf{L}$, and therefore $L \models x \vee x^* \approx 1$, and conversely. Thus, $\mathscr{BSH}$ is also defined, modulo $\mathscr{SH}$, by the identity: $x \vee x^* \approx 1$.

Theorem 9.1. *Let $\mathbf{L} \in \mathscr{SH}$. Then $\mathbf{L}$ is Boolean iff $\mathbf{L} \models x \to y \leq x^* \vee y$.*

Proof. Let $\mathbf{L}$ be Boolean and let $a, b \in L$. Then by Theorem 3.7 (or Theorem 3.8) we get $a \to b \leq a^* \sqcup b = a^* \vee b$. Choose $x = y$ to prove the converse. □

An interesting application of Theorem 6.5 is the following.

Theorem 9.2. $\mathscr{BSH} = V(\mathbf{2},\ \bar{\mathbf{2}})$.

Proof. It is clear that $V(\mathbf{2},\ \overline{\mathbf{2}}) \models x^{**} \approx x$. Since $\mathscr{BSH}$ is the subvariety of $\mathscr{SH}$ defined by the identity: $x^{**} \approx x$, it is immediate that for $L \in \mathscr{BSH}$, Cen $\mathbf{L} = B(\mathbf{L}) = \mathbf{L}$. Hence it follows from Theorem 6.5 that the only directly indecomposable algebras in $\mathscr{V}$ are $\mathbf{2}$ and $\overline{\mathbf{2}}$, and hence $\mathscr{BSH} \subseteq V(\mathbf{2},\ \overline{\mathbf{2}})$, completing the proof. $\square$

The above theorem says that the identity $x \vee x^* \approx 1$ is an equational basis for $V(\mathbf{2},\ \overline{\mathbf{2}})$ modulo $\mathscr{SH}$. Now we give equational bases for $V(\mathbf{2})$ and $V(\overline{\mathbf{2}})$ also. The following corollaries follow immediately from Theorem 9.1.

Corollary 9.3. *An equational basis for $V(\mathbf{2})$, modulo $\mathscr{SH}$, is given by*
(i) $x \vee x^* \approx 1$
(ii) $0 \to 1 \approx 1$.

Corollary 9.4. *An equational basis for $V(\overline{\mathbf{2}})$, modulo $\mathscr{SH}$, is given by*
(i) $x \vee x^* \approx 1$
(ii) $0 \to 1 \approx 0$.

The following corollaries follow from Theorems 9.1 and 9.2.

Corollary 9.5. *An equational basis for $V(\mathbf{2},\overline{\mathbf{2}})$, modulo $\mathscr{SH}$, is: $x \to y \leq x^* \vee y$.*

Corollary 9.6. *An equational basis for $V(\mathbf{2})$, modulo $\mathscr{SH}$, is: $x \to y \approx x^* \vee y$.*

Theorem 9.7. *Another basis for $V(\mathbf{2},\overline{\mathbf{2}})$ is given, relative to $\mathscr{SH}$, by $x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$.*

Proof. Observe that this identity implies $x \vee x^* \approx 1$ (take $x = y$ and $z = 0$). So $x^{**} = x$. Then apply Theorem 9.2. $\square$

Theorem 9.8. *Another basis for $V(\overline{\mathbf{2}})$ is given, relative to $\mathscr{SH}$, by $x \to (y \to z) \approx (x \to y) \to z$* (associative property of $\to$).

Proof. First note that the identity implies $x \to 1 \approx x$ (take $x = y = z$). Also note that the identity implies $x \to 1 \approx x^{**}$ (take $y = 0$, $z = 0$). Hence $x \approx x^{**}$. Also, it is clear that $0 \to 1 \approx 0$. Hence the theorem. $\square$

We know that $\mathscr{SH} \models x \leq (x \to y) \to y$ (see Theorem 2.4(o)). However, the equality in this identity holds only in the variety $V(\overline{\mathbf{2}})$ as shown in the following

Theorem 9.9. *The variety $V(\overline{\mathbf{2}})$ is also defined, modulo $\mathscr{SH}$, by the identity $x \approx (x \to y) \to y$.*

Proof. Observe $\overline{\mathbf{2}}$ satisfies the identity. Now, let $\mathbf{L} \in \mathscr{SH}$ satisfying the identity. Then $\mathbf{L} \models x \approx x^{**}$ (take $y = 0$), hence $\mathbf{L} \in \mathscr{BSH}$. So $\mathbf{L} \cong \mathbf{2}$ or $\mathbf{L} \cong \overline{\mathbf{2}}$. But note that $0 \to 1 = 0$ (take $x = 0$, $y = 1$) in L. Hence $\mathbf{L} \in V(\overline{\mathbf{2}})$. $\square$

We conclude this section by giving a direct proof of a result that $V(\overline{\mathbf{2}})$ is term-equivalent to the variety of Boolean rings.

Theorem 9.10.

(a) *Let $\mathbf{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ be an algebra in $V(\overline{\mathbf{2}})$. Define $\mathbf{L}^{\circledast}$ to be the algebra $\langle L, +^{\circledast}, \cdot^{\circledast}, -^{\circledast}, 0^{\circledast}, 1^{\circledast}\rangle$, where*

$$\begin{array}{rcl} a +^{\circledast} b & = & a \to b \\ a \cdot^{\circledast} b & = & a \vee b \\ -^{\circledast} a & = & a \\ 0^{\circledast} & = & 1 \\ 1^{\circledast} & = & 0 \end{array} .$$

Then $\mathbf{L}^{\circledast}$ is a Boolean ring.

(b) *Let $\mathbf{R} = \langle R, +, \cdot, -, 0, 1\rangle$ be a Boolean ring. Define $\mathbf{R}^{\circledast}$ to be the algebra $\langle R, \vee^{\circledast}, \wedge^{\circledast}, \to^{\circledast}, 0^{\circledast}, 1^{\circledast}\rangle$, where*

$$\begin{array}{rcl} a \vee^{\circledast} b & = & a \cdot b \\ a \wedge^{\circledast} b & = & (a+b) + a \cdot b \\ a \to^{\circledast} b & = & a + b \\ 0^{\circledast} & = & 1 \\ 1^{\circledast} & = & 0. \end{array} .$$

Then $\mathbf{R}^{\circledast}$ is an algebra in $V(\overline{\mathbf{2}})$.

(c) *Given $\mathbf{L}$ and $\mathbf{R}$ as above, then $\mathbf{L}^{\circledast\circledast} = \mathbf{L}$ and $\mathbf{R}^{\circledast\circledast} = \mathbf{R}$.*

Proof. (a) Let $\mathbf{L} \in V(\overline{\mathbf{2}})$.

(i) Commutative law for $+$ holds, since $\overline{\mathbf{2}} \models x \to y \approx y \to x$.
(ii) $0^{\circledast}$ is the additive identity, since $\overline{\mathbf{2}} \models 1 \to x \approx x$ and $\overline{\mathbf{2}} \models x \to 1 \approx x$.
(iii) Associative law for $+$ holds, since $\overline{\mathbf{2}} \models x \to (y \to z) \approx (x \to y) \to z$
(iv) $\mathbf{L}^{\circledast} \models x + x \approx 0$, since $\overline{\mathbf{2}} \models x \to x \approx 1$.
(v) Associative law for $\cdot$ holds, since $\mathbf{L} \models x \vee (y \vee z) \approx (x \vee y) \vee z$.
(vi) Commutative law for $\cdot$ holds, since $\mathbf{L} \models x \vee y \approx y \vee x$.
(vii) $1^{\circledast}$ is the multiplicative identity, since $\mathbf{L} \models x \vee 0 \approx x$.
(viii) $\mathbf{L}^{\circledast} \models x^2 \approx x$, since $\mathbf{L} \models x \vee x \approx x$.
(ix) The distributive law holds in $\mathbf{L}^{\circledast}$ since $\overline{\mathbf{2}} \models x \vee (y \to z) \approx (x \vee y) \to (x \vee z)$.

Thus we conclude that $\mathbf{L}^{\circledast}$ is a Boolean ring.

(b) Let $\mathbf{R}$ be a Boolean ring.

(i) $\mathbf{R}^{\circledast} \models x \vee x \approx x$ since $\mathbf{R} \models x^2 \approx x$.
(ii) Associativity and commutativity of $\vee$ follows from those of $\cdot$.
(iii) Idempotency of $\wedge$: Let $a \in R$. Then $a \wedge a = (a+a) + a \cdot a = 0 + a = a$.
(iv) Associativity of $\wedge$: Let $a, b, c \in R$. Now,

$$\begin{array}{rcl} a \wedge (b \wedge c) & = & a \wedge [(b+c) + b \cdot c] \\ & = & [a + ((b+c) + b \cdot c)] + a \cdot ((b+c) + b \cdot c) \\ & = & a + b + c + a \cdot b + a \cdot c + b \cdot c + a \cdot b \cdot c \\ & = & (a \wedge b) \wedge c. \end{array}$$

(v) Commutativity of $\wedge$ and absorption laws are easy to show.

Hence $\mathbf{R}^{\circledast}$ is a lattice.

(vi) $\mathbf{R}^{\circledast} \models x \to x \approx 1$ since $\mathbf{R} \models x + x \approx 0$.
(vii) For (SH2), let $a, b \in R$. Then

$$\begin{array}{rcl} a \wedge (a \to b) & = & a + a + b + a \cdot (a+b) \\ & = & b + a + a \cdot b, \text{ since } a + b = 0 \text{ and } a^2 = a, \end{array}$$

so $a \wedge (a \to b) = a \wedge b$.

(viii) To prove (SH3), let $a, b, c \in R$. Then

$$\begin{array}{rcl} a \wedge (b \to c) & = & a \wedge (b+c) \\ & = & a+b+c+a\cdot b+a\cdot c. \end{array}$$

Also,

$$\begin{aligned} & a \wedge [(a \wedge b) \to (a \wedge c)] \\ & \quad = a \wedge [(a \wedge b) + (a \wedge c)] \\ & \quad = a + [(a \wedge b) + (a \wedge c)] + a \cdot [(a \wedge b) + (a \wedge c)] \\ & \quad = a + [a+b+a\cdot b+a+c+a\cdot c] + a\cdot [a+b+a\cdot b+a+c+a\cdot c] \\ & \quad = b+a\cdot b+a+c+a\cdot c \end{aligned}$$

Thus (SH3) holds, and hence $\mathbf{R}^{\circledast}$ is a semi-Heyting algebra.

(ix) To prove $\mathbf{R}^{\circledast} \models x \vee x^* \approx 1$, let $a \in R$. Then

$$\begin{array}{rcl} a \vee a^* & = & a \vee (a \to 0^{\circledast}) \\ & = & a \cdot (a+1) \\ & = & a\cdot a+a \\ & = & 0^{\mathbf{R}} \\ & = & 1^{\circledast}. \end{array}$$

(x) $0^{\circledast} \to 1^{\circledast} = 1+0 = 1 = 0^{\circledast}$. Thus $\mathbf{R}^{\circledast} \models 0 \to 1 \approx 0$.

Hence it follows from Corollary 9.4 that $\mathbf{R}^{\circledast} \in V(\overline{\mathbf{2}})$.

Finally, (c) is straightforward to verify and hence is left to the reader. □

10. Stone Semi-Heyting Algebras

In this section we focus our attention on the subvariety $\mathscr{SSH}$ of Stone semi-Heyting algebras of $\mathscr{SH}$. Using the distributivity of $\mathbf{L}$ (see Theorem 3.5), the following theorem gives a characterization of the variety $\mathscr{SSH}$.

Theorem 10.1. *Let $\mathbf{L} \in \mathscr{SH}$. Then $\mathbf{L}$ is Stone iff $\mathbf{L} \models x^{**} \to y \leq x^* \vee y$.*

Proof. Suppose $\mathbf{L}$ is Stone, and let $a, b \in L$. Then

$$\begin{array}{rcl} a^{**} \to b & = & (a^{**} \vee a^*) \wedge (a^{**} \to b) \\ & = & [a^{**} \wedge (a^{**} \to b)] \vee [a^* \wedge (a^{**} \to b)] \\ & = & (a^{**} \wedge b) \vee [a^* \wedge (a^{**} \to b)] \\ & = & [(a^{**} \wedge b) \vee a^*] \wedge [(a^{**} \wedge b) \vee (a^{**} \to b)] \\ & = & (a^* \vee b) \wedge (a^{**} \to b) \text{ by (e) of Theorem 2.4} \end{array}$$

Thus $a^{**} \to b \leq a^* \vee b$. For the converse, observe that $1 = a^{**} \to a^{**} \leq a^* \vee a^{**}$. □

Theorems 6.5 and 7.5, when applied to Stone semi-Heyting algebras, lead us to the following theorem.

Theorem 10.2. *Let $\mathbf{L} \in \mathscr{SSH}$ be subdirectly irreducible. Then*

(*a*) $D(L) = L - \{0\}$,
(*b*) 1 *is* $\vee$*-irreducible.*

Proof. Suppose $\mathbf{L}$ is subdirectly irreducible. Let $a \in L - \{0\}$. If $a^* \neq 0$ then $a^* \vee a^{**} < 1$ by Theorem 6.5(iii), which is impossible since $\mathbf{L} \in \mathscr{SSH}$. Thus $a^* = 0$ and hence $L - \{0\} \subseteq D(L)$, proving (a). (b) follows since $\mathbf{L}$ has a unique coatom in view of Theorem 7.5. □

Applications of the above theorem will be given in Section 11.

11. Equational Bases for 3-valued Semi-Heyting Algebras

The 3-element Heyting algebra has been studied both logically (as a 3-valued logic) and algebraically. In the same vein, the remaining 9 three-element semi-Heyting algebras (see Section 3) can also be viewed as 3-valued logics. The problem of axiomatization of the propositional calculi corresponding to these logics remains open. In this section we investigate them algebraically by looking at their equational theory. In fact, we present (small) equational bases for each of the ten 3-element algebras.

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We denote the height of (the lattice reduct) of L by $h(L)$. For $\mathscr{V}$ a variety, $\mathscr{V}_{SI}$ denotes the class of nontrivial, subdirectly irreducible algebras in $\mathscr{V}$.

The following lemma and Theorem 10.2 will be used frequently in the rest of this section, sometimes without explicit reference to them.

Lemma 11.1. *Let* $\mathbf{L} \in \mathscr{S}\mathscr{H}$ *and* $a, b \in L$ *with* $a < b$. *Then* $b \to a \neq 1$.

Proof. If $b \to a = 1$, then $b = b \wedge 1 = b \wedge (b \to a) = b \wedge a = a$, which is impossible. □

The following identities play a crucial role in the rest of this section.

(I1) $x^* \vee x^{**} \approx 1$ (Stone identity),
(I2) $x \vee (x \to y) \approx (x \to y)^* \to x$,
(I3) $x \vee [y \to (x \vee y)] \approx (0 \to x) \vee (x \to y)$,
(I4) $x \vee (y \to x) \approx [(x \to y) \to y] \to x$,
(I5) $x \vee (x \to y) \approx x \to [x \vee (y \to 1)]$,
(I6) $0 \to 1 \approx 1$ (FTT identity),
(I7) $x \vee (y \to x) \approx (x \vee y) \to x$
(I8) $(x \to y) \to (0 \to y) \approx x \vee [(x \wedge y) \to 1]$,
(I9) $x^* \vee (x \to y) \approx (x \vee y) \to y$,
(I10) $x \vee (0 \to x) \vee (y \to 1) \approx x \vee [(x \to 1) \to (x \to y)]$,
(I11) $x \vee y \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,
(I12) $x \vee [(0 \to y) \to y] \approx x \vee [(x \to 1) \to y]$,
(I13) $x \vee [x \to (y \wedge (0 \to y))] \approx x \to [(x \to y) \to y]$,
(I14) $(0 \to 1)^* = 0$,
(I15) $x \vee y \vee [y \to (y \to x)] \approx x \to [x \vee (0 \to y)]$,
(I16) $x \vee [y \to (0 \to (y \to x))] \approx x \vee y \vee (y \to x)$,
(I17) $x \vee (x \to y) \approx x \vee [(x \to y) \to 1]$,

(I18) $0 \to 1 \approx 0$ (FTF identity),
(I19) $x \vee [y \to (x \vee y^*)] \approx (x \vee y \vee y^*) \to x$,
(I20) $x \to y \approx y \to x$ (commutative identity).

Recall that the variety $\mathscr{SSH}$ of Stone algebras is defined, relative to $\mathscr{SH}$, by (I1). While reading the proofs of the following theorems, it will be helpful to keep in mind that $\mathbf{2}$ is a subalgebra of $\mathbf{L}_i$, for $i = 1,2,3,4$ and is a homomorphic image of $\mathbf{L}_i$, for $i = 5,6,7,8$, and $\bar{\mathbf{2}}$ is a subalgebra of $\mathbf{L}_i$, for $i = 9,10$. It is straightforward to show, in view of Jonsson's Theorem, that $\mathscr{V}(L_i)_{SI} = \{\mathbf{2}, \mathbf{L}_i\}$ for $i = 1,2,\cdots 8$ and $\mathscr{V}(L_i)_{SI} = \{\bar{\mathbf{2}}, \mathbf{L_i}\}$ for $i = 9,10$.

Theorem 11.2. *An equational base for* $\mathbf{L}_1$*, modulo* $\mathscr{SSH}$*, is given by* (I2).

Proof. Clearly $\mathbf{L}_1 \models$ (I2). Now, let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I2). Let $\mathbf{L} \in \mathscr{V}_{SI}$. Then we claim that $h(L) \leq 2$. For, otherwise, let $a,b \in L$ such that $0 < a < b < 1$. Then $a \leq b \to a$ by Theorem 2.4 (f); so, it follows that $b \to a \neq 0$, hence $(b \to a)^* = 0$ by Theorem 10.2. Also, by choosing $x = y = b$ in (I2) we see that $0 \to b = 1$. With $x = b$ and $y = a$ in (I2) we get $b \vee (b \to a) = 0 \to b = 1$. Since 1 is $\vee$-irreducible by Theorem 10.2, it follows that $b \to a = 1$, which is impossible by Lemma 11.1. Thus $h(L) \leq 2$. Now it suffices to show that $\mathbf{L} \cong \mathbf{2}$ or $\mathbf{L} \cong \mathbf{L}_1$. Observe that, with $x = y = 1$, (I2) implies $0 \to 1 = 1$. Hence, if $|L| = 2$ then it is clear that $\mathbf{L} \cong \mathbf{2}$. Next, let $L = \{0,a,1\}$ with $0 < a < 1$. Setting $x = y = a$ in (I2), we get that $0 \to a = 1$. Since $a \leq a \to 1$, we see that $(a \to 1)^* = 0$, so by (I2) we have $a \vee (a \to 1) = 0 \to a = 1$. Hence $a \to 1 = 1$ since 1 is $\vee$-irreducible. Thus $\mathbf{L} \cong \mathbf{L}_1$. So $\mathscr{V} = \mathscr{V}(\mathbf{L}_1)$. □

Theorem 11.3. *A base for* $\mathbf{L}_2$*, modulo* $\mathscr{SSH}$*, is given by* (I3) *and* (I4).

Proof. Let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I3) and (I4). It is easy to see that $\mathbf{L}_3 \in \mathscr{V}$. Let $\mathbf{L} \in \mathscr{V}_{SI}$. We claim that $h(L) \leq 2$. For, suppose $a,b \in L$ such that $0 < a < b < 1$. Then, with $x = a$ and $y = b$, (I3) implies that $1 = (0 \to a) \vee (a \to b)$, whence $0 \to a = 1$ or $a \to b = 1$, as 1 is $\vee$-irreducible. Also, since $a^* = 0$ by Theorem 10.2, it follows from (I4), with $x = a$ and $y = 0$, that $a \vee (0 \to a) = a^{**} \to a = 1 \to a = a$, implying $0 \to a \leq a \neq 1$. Hence, we conclude that $a \to b = 1$. Then by (I3), with $x = b$ and $y = a$, we obtain that $(0 \to b) \vee (b \to a) = 1$, yielding $0 \to b = 1$ or $b \to a = 1$, as 1 is $\vee$-irreducible. But, since $b^* = 0$ by Theorem 10.2, it follows from (I4), with $x = b$ and $y = 0$, that $b \vee (0 \to b) = b^{**} \to b = b$, thus $0 \to b \leq b$. Hence, we conclude that $b \to a = 1$, contradicting Lemma 11.1. Thus $h(L) \leq 2$. Now, with $x = 1$ and $y = 0$, (I3) implies $0 \to 1 = 1$. Hence, if $|L| = 2$, then clearly $\mathbf{L} \cong \mathbf{2}$. So we assume that $|L| = 3$, say $L = \{0,a,1\}$, where $0 < a < 1$. Now, as seen before, with $x = a$ and $y = 0$, (I4) implies $0 \to a \leq a$. Also, from $0 \to 1 = 1$ (proved earlier), we can conclude, using (SH3), that $a \leq 0 \to a$, yielding $0 \to a = a$. Next, with $x = a$ and $y = 1$, (I3) yields $(0 \to a) \vee (a \to 1) = 1$, so $a \vee (a \to 1) = 1$, implying $a \to 1 = 1$. Thus $\mathbf{L} \cong \mathbf{L}_2$ and hence $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L}_2\}$. □

Theorem 11.4. *A base for* $\mathbf{L}_3$*, modulo* $\mathscr{SSH}$*, is given by* (I5), (I6) *and* (I7).

Proof. We first note that (I5), (I6)and (I7) are true in $\mathbf{L}_3$. Let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I5), (I6) and (I7). Let $\mathbf{L} \in \mathscr{V}_{SI}$. We claim that $h(L) \leq 2$. For, let $a,b \in L$ such that $0 < a < b < 1$. Then, since $a^* = 0$, with $x = a$ and $y = 0$ in (I5), and using (I6), we get $a \to 1 = a$. Hence, setting $x = b$ and $y = a$ in (I5), we get $b \vee (b \to a) = 1$, implying $b \to a = 1$, as 1 is $\vee$-irreducible. But this is contrary to Lemma 11.1. Thus, $h(L) \leq 2$. Since $0 \to 1 = 1$, $|L| = 2$ would clearly imply $\mathbf{L} \cong \mathbf{2}$. Thus we may assume that $L = \{0,a,1\}$

with $0 < a < 1$. Now we already know $0 \to 1 = 1$, and we can argue as before, using (I5) and (I6), to conclude that $a \to 1 = a$. Also, from (I7), with $x = a$ and $y = 0$, we get $a \vee (0 \to a) = 1$, implying $0 \to a = 1$. Thus $\mathbf{L} \cong \mathbf{L}_3$, proving that $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L}_3\}$, which completes the proof. □

Theorem 11.5. *A base for* $\mathbf{L}_4$*, modulo* $\mathscr{SSH}$*, is defined by* (I5)*,* (I6) *and* (I8)*.*

Proof. Let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I5), (I6) and (I8) and let $\mathbf{L} \in \mathscr{V}_{SI}$. Then since (I5) and (I6) are true in $\mathbf{L}$, it can be easily shown, as in the proof of Theorem 11.4, that $h(L) \leq 2$. We may suppose that $\mathbf{L} = \{0, a, 1\}$ with $0 < a < 1$. We obtain $0 \to 1 = 1$ and $a \to 1 = a$ as in the proof of Theorem 11.4. Now, from (I8) we get $(a \to a) \to (0 \to a) = a \vee (a \to 1)$ implying $0 \to a = a$, since $a \to 1 = a$. Thus we conclude that $\mathbf{L} \cong \mathbf{L}_4$ and $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L}_4\}$. □

Theorem 11.6. *A base for* $\mathbf{L}_5$*, modulo* $\mathscr{SSH}$*, is given by* (I9) *and* (I10)*.*

Proof. Let $\mathbf{L} \in \mathscr{V}_{SI}$, where $\mathscr{V}$ is the subvariety of $\mathscr{SSH}$ defined by (I9) and (I10). Suppose $0 < a < b < 1$ in $\mathbf{L}$. Since $a^* = 0 = b^*$, (I9), with $x = b$ and $y = 1$, implies that $b \to 1 = 1$; and similarly we conclude $a \to 1 = 1$. Hence, from (I10), with $x = b$ and $y = a$, we get $1 = b \vee (b \to a)$, which implies $b \to a = 1$, which is a contradiction in view of Lemma 11.1. Hence $h(L) \leq 2$. Setting $x = y = 0$ in (I10) we get that $(0 \to 1) \to 1 = 1$, so $0 \to 1 \neq 0$. Now, If $|L| = 2$ then clearly $0 \to 1 = 1$, hence $\mathbf{L} \cong \mathbf{2}$. Next, suppose $\mathbf{L} = \{0, a, 1\}$ with $0 < a < 1$. With $x = a$ and $y = 0$, (I10) implies that $a \vee (0 \to a) \vee (0 \to 1) = a$, implying $0 \to a \leq a$ and $0 \to 1 \leq a$. Since $0 \to 1 \neq 0$, we conclude that $0 \to 1 = a$. Therefore it follows from Theorem 2.4(r) that $a \leq 0 \to a$, thus we conclude that $0 \to a = a$. We also have $a \to 1 = 1$ as shown before. Hence $\mathbf{L} \cong \mathbf{L}_5$ and $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L_5}\}$. □

Theorem 11.7. *A base for* $\mathbf{L}_6$*, modulo* $\mathscr{SSH}$*, is given by* $\{(\text{I9}), (\text{I11})\}$*.*

Proof. Let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I9) and (I11) and $\mathbf{L} \in \mathscr{V}_{SI}$. Suppose $0 < a < b < 1$ in $\mathbf{L}$. Since $a \leq b \to a$, we have $b \to a \neq 0$. Hence, $(b \to a)^* = 0$. So by (I9), with $x = b \to a$ and $y = 1$, we have $[(b \to a) \to 1] = [(b \to a) \vee 1] \to 1 = 1 \to 1 = 1$. Thus we conclude that $(b \to a) \to 1 = 1$. Then by (I11), setting $x = b$ and $y = a$, we get $b \vee (b \to a) = 1$, implying $b \to a = 1$, leading to a contradiction. Thus $h(L) \leq 2$. Observe that, with $x = 0$ and $y = 1$, (I11) implies $(0 \to 1) \to 1 = 1$. Hence $0 \to 1 \neq 0$. If $|L| = 2$, then we conclude $0 \to 1 = 1$, and $\mathbf{L} \cong \mathbf{2}$. So, let $L = \{0, a, 1\}$ with $0 < a < 1$. Since $a^* = 0$, we infer from (I9), using $x = a$ and $y = 1$, that $a \to 1 = 1$. Also, from (I11), with $x = a$ and $y = 0$, we get $a = a \vee (0 \to 1)$, so $0 \to 1 \leq a$. But we know that $0 \to 1 \neq 0$, so $0 \to 1 = a$. Then by Theorem 2.4(r) we have $a \leq 0 \to a$. We claim that $0 \to a = 1$. For, if $0 \to a = a$, then from (I11), with $x = 0$ and $y = a$, we get $a = a \vee (0 \to a) = (0 \to a) \to 1 = a \to 1 = 1$; so we have a contradiction, proving the claim. Hence $\mathbf{L} \cong \mathbf{L}_6$, and $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L_6}\}$. □

Theorem 11.8. *A base for* $\mathbf{L}_7$*, modulo* $\mathscr{SSH}$*, is given by* (I12) *and* (I13)*.*

Proof. Let $\mathscr{V}$ be the subvariety of $\mathscr{SSH}$ defined by (I12) and (I13). Let $\mathbf{L} \in \mathscr{V}_{SI}$. Suppose $0 < a < b < 1$ in L. (I13), with $x = a$ and $y = 0$, gives $a \to 1 = a$. By a similar argument we can conclude that $b \to 1 = b$. Then from (I12), with $x = y = a$, we get $a \vee [(0 \to a) \to a] = a \vee [(a \to 1) \to a] = a \vee (a \to a) = 1$. Thus $(0 \to a) \to a = 1$, since 1 is $\vee$-irreducible. Hence, it follows from (I12), with $x = b$ and $y = a$, that $b \vee (b \to a) = 1$. Then $b \to a = 1$, which is impossible. Thus we conclude that $h(L) \leq 2$. Observe from (I12), with $x = 0$ and $y = 0$, that $(0 \to 1)^* = 0$, so $0 \to 1 \neq 0$. Hence, if $|L| = 2$ then $0 \to 1 = 1$, implying

$\mathbf{L} \cong \mathbf{2}$. So, suppose $L = \{0, a, 1\}$ with $0 < a < 1$. Now we can conclude $a \to 1 = a$ and $(0 \to a) \to a = 1$ as before. Then, with $x = 0$ and $y = a$, (I12) gives $(0 \to 1) \to a = 1$, from which it follows that $0 \to 1 \neq 1$. Also, we know that $0 \to 1 \neq 0$. Hence from Theorem 2.4(r) we get $a \leq 0 \to a$. Observe that $0 \to a = 1$ is impossible since $(0 \to a) \to a = 1$. Thus $0 \to a = a$, from which we conclude that $\mathbf{L} \cong \mathbf{L}_7$, and $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L_7}\}$. □

Theorem 11.9. *A base for* $\mathbf{L}_8$*, relative to* $\mathscr{S}\mathscr{S}\mathscr{H}$*, is given by* (I14), (I15) *and* (I16).

Proof. Let $\mathbf{L} \in \mathscr{V}_{SI}$, where $\mathscr{V} \subseteq \mathscr{S}\mathscr{S}\mathscr{H}$ is defined by (I14), (I15) and (I16). Let $0 < a < b < 1$ in $\mathbf{L}$. Then $a^* = b^* = 0$. Letting $x = b$ and $y = 1$ in (I16), we conclude that $0 \to b = 1$. Similarly, we have $0 \to a = 1$. With $x = 0$ and $y = b$, (I15) implies $b = 0 \to (0 \to b)$, implying that $b = 0 \to 1$, since $0 \to b = 1$. Similarly, we get $a = 0 \to (0 \to a)$, whence $a = 0 \to 1$. Then we have $b = 0 \to 1 = a$, a contradiction. Thus $h(L) \leq 2$. Now, if $|L| = 2$, then $0 \to 1 \neq 0$ in view of (I14), so $0 \to 1 = 1$, whence $\mathbf{L} \cong \mathbf{2}$. So, let $L = \{0, a, 1\}$ with $0 < a < 1$. Then, as argued before, (I16) implies $0 \to a = 1$. Since (I15) implies $0 \to (0 \to a) = a$ as shown before, we get $a = 0 \to (0 \to a) = 0 \to 1$. Finally, with $x = 0$ and $y = a$, (I16) yields $a \to 1 = a$, implying $\mathbf{L} \cong \mathbf{L}_8$, and $\mathscr{V}_{SI} = \{\mathbf{2}, \mathbf{L_8}\}$. □

Theorem 11.10. *A base for* $\mathbf{L}_9$*, modulo* $\mathscr{S}\mathscr{S}\mathscr{H}$*, is given by* (I9), (I17) *and* (I18).

Proof. Let $\mathscr{V}$ be a subvariety of $\mathscr{S}\mathscr{S}\mathscr{H}$ defined by (I9), (I17) and (I18). Let $\mathbf{L} \in \mathscr{V}_{SI}$ and $a, b \in L$ such that $0 < a < b < 1$. Since $a \leq b \to a$, we have $b \to a \neq 0$. Then it follows from (I9), with $x = b \to a$ and $y = 1$, that $(b \to a) \to 1 = 1$ as $(b \to a)^* = 0$. Then (I17), with $x = b$ and $y = a$ yields $b \vee (b \to a) = b \vee [(b \to a) \to 1] = 1$, implying that $b \to a = 1$, which is impossible; hence $h(L) \leq 2$. Since $0 \to 1 = 0$, if $|L| = 2$, then clearly $\mathbf{L} \cong \bar{\mathbf{2}}$. So, suppose $L = \{0, a, 1\}$ with $0 < a < 1$. Since $a^* = 0$, it follows from (I9), with $x = a$ and $y = 1$, that $a \to 1 = 1$. Next, since $0 \to 1 = 0$, we get $a \wedge (0 \to a) = 0$ by (SH3), from which we conclude that $0 \to a = 0$ since 0 is $\wedge$-irreducible. Thus $\mathbf{L} \cong \mathbf{L}_9$, and $\mathscr{V}_{SI} = \{\bar{\mathbf{2}}, \mathbf{L_9}\}$. □

Theorem 11.11. *A base for* $\mathbf{L}_{10}$*, relative to* $\mathscr{S}\mathscr{S}\mathscr{H}$*, is given by* (I19) *and* (I20).

Proof. Let $\mathscr{V}$ be a subvariety of $\mathscr{S}\mathscr{S}\mathscr{H}$ defined by (I19) and (I20). Let $\mathbf{L} \in \mathscr{V}_{SI}$ and $a, b \in L$ with $0 < a < b < 1$. Since $a^* = 0$, we obtain from (I19), with $x = b$ and $y = a$, that $b \vee (a \to b) = b \to b = 1$. Hence $a \to b = 1$. Then (I20) would imply $b \to a = 1$, leading to a contradiction. Thus $h(L) \leq 2$. Observe that (I20) implies $0 \to 1 = 0$. Then, if $|L| = 2$, then it is obvious that $\mathbf{L} \cong \bar{\mathbf{2}}$. So, let $L = \{0, a, 1\}$ with $0 < a < 1$. Now, it follows from (I20) that $0 \to a = a^* = 0$ and $a \to 1 = 1 \to a = a$. So $\mathbf{L} \cong \mathbf{L}_{10}$, and $\mathscr{V}_{SI} = \{\bar{\mathbf{2}}, \mathbf{L_{10}}\}$. □

We conclude this section by mentioning that the identity (I1) is redundant in the bases mentioned in all the above theorems, except in the Theorem 11.9; but the proofs will appear in [33].

12. New Bases for Heyting Algebras

An interesting by-product of our investigations into semi-Heyting algebras is the discovery of several new axiom systems for the variety of Heyting algebras. The following theorem gives several new axiom systems for the variety of Heyting algebras by augmenting the axioms (see Definition 2.3) of $\mathscr{S}\mathscr{H}$ with a single new axiom in each case.

Theorem 12.1. *Let* $\mathbf{L}$ *be a semi-Heyting algebra. Then the following are equivalent:*

(a) $\mathbf{L}$ *is a Heyting algebra.*
(b) $x \le y$ *implies* $x \to y = 1$
(c) $x \le y$ *implies* $x \to z \ge y \to z$
(d) $x \le y$ *implies* $z \to x \le z \to y$
(e) $(x \vee y) \to z \le (x \to z) \wedge (y \to z)$
(f) $x \to (y \wedge z) \le (x \to y) \wedge (x \to z)$
(g) $x \to (y \to z) \approx (x \wedge y) \to z$
(h) $(x \vee y) \to y \le x \to y$
(i) $[(x \to y) \wedge (y \to z)] \to (x \to z) \approx 1$
(j) $(x \to y) \ge (x^* \vee y)^{**}$
(k) $x \to y \approx x \to (x \wedge y)$.

Proof. It is straightforward to verify that (a) implies each of the remaining conditions. Since (b) implies $(x \wedge y) \to x \approx 1$, it is immediate that (b) $\Rightarrow$ (a) (see Definition 2.1). Let $a \le b$. Then we get from (c) that $a \to b \ge b \to b = 1$, and hence (c) $\Rightarrow$ (b). From (d) and $x \le y$ we have $1 = x \to x \le x \to y$, so that $x \to y = 1$, and so (d) $\Rightarrow$ (b). From (e) we have $1 = z \to z = ((x \wedge z) \vee z) \to z \le ((x \wedge z) \to z) \wedge (z \to z) = (x \wedge z) \to z$, so (a) holds, thus (e) $\Rightarrow$ (a). From (f) it follows that $1 = (y \wedge z) \to (y \wedge z) \le [(y \wedge z) \to y] \wedge [(y \wedge z) \to z]$, so $(y \wedge z) \to y = 1$, yielding (f) $\Rightarrow$ (a). To prove (g) $\Rightarrow$ (a), first we note that (g) implies $x \to 1 = 1$, since $x \to (0 \to 0) = 0 \to 0$. Also from (g) we have $1 = x \to 1 = x \to (y \to y) = (x \wedge y) \to y$, thus (a) holds. Next, from (h) we get $((x \wedge y) \vee y) \to y \le (x \wedge y) \to y$, which implies $y \to y \le (x \wedge y) \to y$, hence $(x \wedge y) \to y = 1$, so (a) holds. For (i)$\Rightarrow$ (a), take $x = 1$ and use (SH2).

From (j) we have

$$\begin{aligned} (x \wedge y) \to y &\ge [(x \wedge y)^* \vee y]^{**} \\ &= [(x^* \vee y^*)^{**} \vee y]^{**} \\ &= [(x^* \vee y^*)^* \wedge y^*]^* \\ &= [x^{**} \wedge y^{**} \wedge y^*]^* \\ &= 0^* \\ &= 1. \end{aligned}$$

Thus $(x \wedge y) \to y = 1$, so (a) holds, giving (j) $\Rightarrow$ (a). To prove (k) $\Rightarrow$ (a), replace x by $x \wedge y$ in (k) and use the identity (SH4). $\square$

In the next two theorems we give axiomatizations of Heyting algebras that do not include the axioms of semi-Heyting algebras.

In the rest of this section, let $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1 \rangle$ denote an algebra such that $\langle A, \vee, \wedge, 0, 1 \rangle$ is a lattice with $0, 1$ and $\to$ is a binary operation.

Theorem 12.2. $\mathbf{A}$ *is a Heyting algebra iff* $\mathbf{A}$ *satisfies the following conditions:*

(i) $x \wedge (x \to y) \le y$
(ii) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$
(iii) $(x \wedge y) \to y \approx 1$.

Proof. Suppose (i) - (iii) hold in $\mathbf{A}$. Now $y \wedge (x \to y) = y \wedge [(y \wedge x) \to y] = y$ by (ii) and (iii), so that $y \le x \to y$. Hence $x \wedge y \le x \wedge (x \to y)$. Also, (i) implies $x \wedge (x \to y) \le x \wedge y$, thus $x \wedge (x \to y) = x \wedge y$. It follows that $\mathbf{A}$ is a Heyting algebra (see Definition 2.1). Since the converse is well known, the proof is complete. $\square$

Theorem 12.3. *The following are equivalent:*

(a) **A** *is a Heyting algebra.*

(b) **A** *satisfies:*

- (i) $x \wedge [(x \wedge y) \to z] \leq x \wedge (y \to z)$
- (ii) $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$
- (iii) $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$
- (iv) $x \to x \approx 1$
- (v) $1 \to x \approx x.$

Proof. Suppose (b) holds. Then using (iii) we have

$$\begin{aligned} x \wedge (y \to z) &\approx x \wedge [((x \wedge y) \vee y) \to z] \\ &\approx x \wedge [(x \wedge y) \to z] \wedge (y \to z), \end{aligned}$$

so $x \wedge (y \to z) \leq x \wedge [(x \wedge y) \to z]$, which, combined with (i), yields $x \wedge [(x \wedge y) \to z] \approx x \wedge (y \to z)$. Also, from (iv) and (iii) we have

$$1 \approx z \to z \approx ((x \wedge z) \vee z) \to z \leq (x \wedge z) \to z,$$

thus $(x \wedge z) \to z \approx 1$. Next, using (ii) and the two identities just proved, we get

$$\begin{aligned} x \wedge [(x \wedge y) \to (x \wedge z)] &\approx x \wedge [(x \wedge y) \to x] \wedge [(x \wedge y) \to z] \\ &\approx x \wedge [(x \wedge y) \to z] \\ &\approx x \wedge (y \to z). \end{aligned}$$

Finally, (i) and (v) imply

$$\begin{aligned} x \wedge (x \to y) &\leq x \wedge (1 \to y) \text{ by (i)} \\ &\approx x \wedge y \text{ by (v)} \\ &\leq y. \end{aligned}$$

Thus it follows from the previous theorem that **A** is a Heyting algebra, hence, (b) $\Rightarrow$ (a). Since the converse is well known, the proof is complete. □

We conjecture that (b)(ii) in the preceding theorem is redundant. The following theorem is well known (see [16]).

Theorem 12.4. *The following are equivalent:*

(a) **A** *is a Heyting algebra.*

(b) **A** *satisfies:*

- (i) $x \to x \approx 1$
- (ii) $y \wedge (x \to y) \approx y$
- (iii) $x \wedge (x \to y) \approx x \wedge y$
- (iv) $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z).$

Theorem 12.5. *The following are equivalent:*

(a) **A** *is a Heyting algebra.*

(b) **A** *satisfies:*

- (i) $x \wedge (x \to y) \approx x \wedge y$
- (ii) $y \wedge (x \to y) \approx y$
- (iii) $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$
- (iv) $(x \wedge y) \to x \approx 1$
- (v) $(x \to y) \wedge (t \to s) \approx (x \to y) \wedge [(t \wedge (x \to y)) \to (s \wedge (x \to y))].$

Proof. Suppose (b) holds. Then

$(x \to y) \wedge (x \to z) \wedge [x \to (y \wedge z)]$

$$
\begin{aligned}
&\approx \quad (x \to y) \wedge (x \to z) \wedge [(x \wedge (x \to y)) \to (y \wedge z \wedge (x \to y))] \text{ by (v)}\\
&\approx \quad (x \to y) \wedge (x \to z) \wedge [(x \wedge y) \to (y \wedge z)] \text{ by (i) and (ii)}\\
&\approx \quad (x \to y) \wedge (x \to z) \wedge [(x \wedge y \wedge (x \to z)) \to ((x \to z) \wedge y \wedge z))] \text{ by (v)} \quad \text{Thus } x \to\\
&\approx \quad (x \to y) \wedge (x \to z) \wedge [(x \wedge y \wedge z) \to (y \wedge z)] \text{ by (i) and (ii)}\\
&\approx \quad (x \to y) \wedge (x \to z) \text{ by (iv)} .
\end{aligned}
$$

$(y \wedge z) \geq (x \to y) \wedge (x \to z)$, which, together with (iii), implies (iv) of Theorem 12.4. Also, (iv) implies $x \to x \approx 1$. Hence in view of Theorem 12.4 we conclude that (a) holds. □

13. Semi-Brouwerian Algebras, Semi-Heyting Semilattices and Semi-Brouwerian Semilattices

The observation that the axioms (SH2), (SH3) and (SH4) of Definition 2.3 contain neither the constant 0 nor the operation $\vee$ leads us to consider three new (equational) classes of algebras mentioned in the title of this section. We now give their definitions and point out which of the results proved in the previous sections remain true for these classes.

Definition 13.1. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \to, 1 \rangle$ is a *semi-Brouwerian algebra*, if

(SB1) $\langle L, \vee, \wedge, 1 \rangle$ is a lattice with 1,
(SB2) $x \wedge (x \to y) \approx x \wedge y$,
(SB3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,
(SB4) $x \to x \approx 1$.

The variety of these algebras is denoted by $\mathscr{SB}$.

Definition 13.2. An algebra $\mathbf{L} = \langle L, \wedge, \to, 0, 1 \rangle$ is a *semi-Heyting semilattice*, if

(SHS1) $\langle L, \wedge, 0, 1 \rangle$ is a semilattice with $0, 1$,
(SHS2) $x \wedge (x \to y) \approx x \wedge y$,
(SHS3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,
(SHS4) $x \to x \approx 1$.

We denote by $\mathscr{SHS}$ the variety of semi-Heyting semilattices.

Definition 13.3. An algebra $\mathbf{S} = \langle S, \wedge, \to, 1 \rangle$ is a *semi-Brouwerian semilattice*, if

(SBS1) $\langle S, \wedge, 1 \rangle$ is a semilattice with 1,
(SBS2) $x \wedge (x \to y) \approx x \wedge y$,
(SBS3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$,
(SBS4) $x \to x \approx 1$.

The variety of these algebras is denoted by $\mathscr{SBS}$. One can also define the duals to the above three classes, as well as combinations of these, such as double semi-Brouwerian algebras, etc., which are investigated in [34].

The following table indicates which of the results and definitions of the previous sections hold in the varieties $\mathscr{SB}, \mathscr{SHS}$, and $\mathscr{SBS}$ after obvious modifications in their wording, such as changing "lattice" to "semilattice" (and in the type, such as deleting 0 or $\vee$) or in the proofs.

Number	$\mathcal{SB}$	$\mathcal{SHS}$	$\mathcal{SBS}$	Number	$\mathcal{SB}$	$\mathcal{SHS}$	$\mathcal{SBS}$
2.4	✓	✓	✓	4.1	✓	✓	✓
2.6	✓	✓	✓	4.2	✓except (ii)	✓	✓except (ii)
2.7	✓	✓	✓	4.3	✓	✓	✓
2.8	✓	✓	✓	4.4	✓	✓	✓
2.9	✓	✓	✓	4.5	✓	✓	✓
2.10	✓	✓	✓	4.6	✓	✓	✓
3.1	✓	✓	✓	4.7	✓	✓	✓
3.2	-	✓	-	4.8	✓	✓	✓
3.3	✓	?	?	5.1	✓	✓	✓
3.4	-	✓	-	5.2	✓	✓	✓
3.5	✓	?	?	5.3	✓	✓	✓
3.6	✓	✓	✓	5.4	✓	✓	✓
3.7	-	✓	-	5.6	✓	✓	✓
3.8	-	-	-	5.7	✓	✓	✓
3.9	-	✓	-	5.8	✓	?	?

Number	$\mathcal{SB}$	$\mathcal{SHS}$	$\mathcal{SBS}$	Number	$\mathcal{SB}$	$\mathcal{SHS}$	$\mathcal{SBS}$
6.1	-	✓	-	7.6	✓	✓	✓
6.2	-	✓	-	8.1	✓	✓	✓
6.3	-	✓	-	8.2	-	✓	-
6.4	-	?	-	8.3	-	✓	-
6.5	?	?	?	8.4	-	✓	-
6.6	?	?	?	8.5	✓	✓	✓
7.1	✓	✓	✓	8.6	✓	✓	✓
7.2	✓	✓	✓	8.6	✓	✓	✓
7.3	?	?	?	8.7	✓	✓	✓
7.4	?	?	?	9.2	?	✓	?
7.5	✓	✓	✓	9.8	✓	✓	✓
				9.9	?	✓	?

It should be noted here that the parts (r) and (s) of Theorem 2.4 would only hold for $\mathcal{SB}$ and $\mathcal{SBS}$ if 0 exists in each member of these varieties and Corollary 3.4 holds in $\mathcal{SHS}$ except for (xiv). $\mathcal{SSH}$ should be left out when using Definition 8.1 in $\mathcal{SB}$ and $\mathcal{SBS}$. We also note that Theorem 5.8 holds in the subvariety of $\mathcal{SHS}$ (and also in the subvariety of $\mathcal{SBS}$) defined by $x \leq (x \to y) \to y$.

14. Open Problems

Problem 14.1. Find a duality for $\mathcal{SH}$ or for any of its subvarieties (other than $\mathcal{H}$) mentioned in Section 8 (similar to Priestley's duality for $\mathcal{H}$, for instance).

Problem 14.2. Investigate the structure of the lattice of subvarieties of $\mathscr{SH}$ or any of its subvarieties (mentioned in Section 8). The lattice of subvarieties of $\mathscr{H}$ has received considerable attention.

Problem 14.3. Axiomatize the three-valued semi-Heyting logics $L_2 - L_{10}$ (see figure 3) from the logical point of view.

Problem 14.4. It is known that there are $2^{\aleph_0}$ subvarieties of $\mathscr{H}$. Is the same true for the subvariety of $\mathscr{SH}$ defined by $0 \to 1 \approx 0$?

Problem 14.5. Is the first-order theory of the subvariety $\mathscr{CSH}$ of $\mathscr{SH}$ generated by the semi-Heyting chains decidable ?

Problem 14.6. Find an equational basis for the variety $\mathscr{CSH}$ of semi-Heyting chains.

Problem 14.7. Find an equational basis for the variety $\mathscr{CFTT}$.

Problem 14.8. Find an equational basis for the variety $\mathscr{CQH}$ of quasi-Heyting chains.

Problem 14.9. Find an equational basis for the variety $\mathscr{CFTF}$.

Problem 14.10. Find an equational basis for the variety $\mathscr{C}\mathrm{com}\mathscr{SH}$ of commutative semi-Heyting chains.

Problem 14.11. We proved in Section 13 that $\mathscr{CPTP} = \mathscr{C}\mathrm{com}\mathscr{SH}$. It is also clear that $\mathrm{com}\mathscr{SH} \subseteq \mathscr{PTP}$. Is it true that $\mathscr{PTP} = \mathrm{com}\mathscr{SH}$?

Problem 14.12. For $\mathscr{V}$ a subvariety of $\mathscr{SH}$ and n a natural number, $f(\mathscr{V},n)$ denotes the number of non-isomorphic semi-Heyting chains in $\mathscr{V}$. Find a formula for $f(\mathscr{V},n)$. We know from Sections 2 and 4 that $f(\mathscr{SH},2) = 2$, $f(\mathscr{SH},3) = 10$ and $f(\mathscr{SH},4) = 160$.

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