ACTAS DEL IX CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2007), Páginas 127–129

ON $\mathscr{B}$-OPERATOR DERIVATIVES ON NON AMENABLE NUCLEAR BANACH ALGEBRAS

C. C. PEÑA

ABSTRACT. We review recent advances and some problems related to our research about bounded derivations on non amenable nuclear Banach algebras.

Let $\mathfrak{X}$ be an infinite dimensional complex Banach space. By $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ we will denote the completion of the algebraic tensor product of $\mathfrak{X}$ and $\mathfrak{X}^*$ with respect to the projective cross norm $\|\circ\|_\pi$. Thus $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ becomes a Banach algebra by means of the product so that $(x\otimes x^*)(y\otimes y^*) = \langle y,x^*\rangle (x\otimes y^*)$ if $x,y\in\mathfrak{X}$, $x^*,y^*\in\mathfrak{X}^*$. Let $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ be the subclass of nuclear operators of $\mathscr{B}(\mathfrak{X})$. All $T\in\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ can be writen as $Tx=\sum_{n=1}^\infty \langle x,y_n^*\rangle y_n$ if $x\in\mathfrak{X}$, with $\{y_n\}_{n=1}^\infty\subseteq\mathfrak{X}$, $\{y_n^*\}_{n=1}^\infty\subseteq\mathfrak{X}^*$ and $\sum_{n=1}^\infty \|y_n\|\,\|y_n^*\|<\infty$. The infimum of these series taking over all such representations of T furnish a norm $\|T\|_{\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})}$ for T so that $\left(\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X}), \|\circ\|_{\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})}\right)$ becomes a Banach algebra.

Amenable Banach algebras were introduced and studied by B. E. Johnson in his definitive monograph [5]. Particularly, the notion of amenability is closely related with questions concerning to bounded derivations on Banach algebras. Briefly, a Banach algebra $\mathscr{U}$ is called *amenable* if its first Hochschild cohomology group $H^1(\mathscr{U},X^*)$ with coefficients in the dual of any Banach $\mathscr{U}$-bimodule X is trivial. If this is the case any derivation $D:\mathscr{U}\to X^*$ is *inner*, i.e. there exists $\lambda\in X^*$ so that $D(a)=\lambda\cdot a - a\cdot\lambda$ if $a\in\mathscr{U}$. Indeed, $\mathscr{U}$ is called *super-amenable* when the first cohomology group of $\mathscr{U}$ with coefficients in any Banach $\mathscr{U}$-bimodule is trivial.

Theorem 1. *(cf.* [8]*, Th. 4.3.5, p. 98) The following assertions are equivalent*

- i $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *is super-amenable.*
- ii $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *is amenable.*
- iii $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *has a bounded approximate identity.*
- iv $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *has a bounded left approximate identity.*
- v $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ *has a bounded left approximate identity.*
- vi $\dim(\mathfrak{X})=\dim(\mathfrak{X}^*)<\infty$.

Consequently, the study of bounded derivations on $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ has its own interest as well as the determination of their structure and properties. Fortunately, there is an isometric isomorphism of Banach algebras between $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ and $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ (cf. [8], Th. C.1.5). This fact allowed us to improve previous researches done in the frame of Banach algebras of Hilbert-Schmidt type (cf. [1], [2]). The class of bounded derivations $\mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ is a Banach subspace of $\mathscr{B}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.

2000 *Mathematics Subject Classification.* 46H20, 46H25.
Key words and phrases. Bounded shrinking basis, Associated sequence of coefficient functionals, Projective cross norm, Amenable and super-amenable Banach algebras, Multiplier operator.

Example 2. *Let* $\upsilon \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$*,* $\Delta_\upsilon(\alpha) = \upsilon\cdot\alpha - \alpha\cdot\upsilon$*,* $\alpha \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$*. Therefore* $\Delta_\upsilon \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ *is the inner derivation defined by* υ*. In general, it is known that every bounded derivation on the uniform Banach algebra of bounded operators* $\mathscr{B}(\mathfrak{X})$ *is inner (cf.* [6]*).*

Problem 3. *What is the precise norm of* Δ_υ*?- This problem could be hard. For instance, let* $\mathfrak{X}$ *be a Hilbert space,* $T \in \mathscr{B}(\mathfrak{X})$*,* Δ_T *be the inner derivation induced by* T *on* $\mathscr{B}(\mathfrak{X})$*. Then J. G. Stampfli showed that* $\|\Delta_T\| = 2\,\mathrm{dist}(T, \mathbb{C}\cdot \mathrm{Id}_{\mathfrak{X}})$ *(cf.* [11]*). B. E. Johnson noted that the above formula is no longer true in the general case. If* $\mathfrak{X}$ *is a uniformly convex Banach space the validity of Stampfli´s formula is a necessary and sufficient condition in order that* $\mathfrak{X}$ *be a Hilbert space (see* [4] *and* [7]*).*

Example 4. *Given* $T \in \mathscr{B}(\mathfrak{X})$ *there is a unique* $\delta_T \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ *so that*

$$\delta_T(x\otimes x^*) = T(x)\otimes x^* - x\otimes T^*(x^*)$$

for all basic tensor $x\otimes x^* \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$*. It is said that* δ_T *is the* $\mathscr{B}$*-derivation supported by* T*.*

Problem 5. *Let* $\delta : \mathscr{B}(\mathfrak{X}) \to \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$*,* $\delta(T) = \delta_T$ *if* $T \in \mathscr{B}(\mathfrak{X})$*. Then* δ *is a linear bounded operator so that*

$$\delta(S\circ T) \triangleq [\delta(S), \delta(T)] = \delta(S)\circ\delta(T) - \delta(T)\circ\delta(S)$$

if $S, T \in \mathscr{B}(\mathfrak{X})$*. It would be relevant to evaluate* $\|\delta\|$*.*

Lemma 6. $\ker(\delta) = \mathbb{C}\cdot\mathrm{Id}_{\mathfrak{X}}$.

Proof. Let $T \in \mathscr{B}(\mathfrak{X})$ so that $\delta_T = 0$ and let $\lambda \in \sigma(T)$. If λ belongs to the compression spectrum of T let $x^* \in \mathfrak{X}^* - \{0\}$ so that $x^*|_{\mathrm{R}(T-\lambda\,\mathrm{Id}_{\mathfrak{X}})} \equiv 0$. For all $x \in \mathfrak{X}$ we have

$$\langle x, T^*(x^*)\rangle = \langle T(x), x^*\rangle = \langle \lambda x, x^*\rangle = \langle x, \lambda x^*\rangle,$$

i.e. $(T^* - \lambda\,\mathrm{Id}_{\mathfrak{X}^*})(x^*) = 0$. Moreover, since

$$(T(x) - \lambda x)\otimes x^* = x\otimes(T^*(x^*) - \lambda x^*) = 0,$$

the projective norm is a cross-norm and $x^* \neq 0$ then $T = \lambda\,\mathrm{Id}_{\mathfrak{X}}$. If $\lambda \in \sigma_{ap}(T)$ we choose a sequence $\{y_n\}_{n=1}^\infty$ of unit vectors of $\mathfrak{X}$ so that $T(y_n) - \lambda y_n \to 0$. If $y^* \in \mathfrak{X}^*$ then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(T(y_n) - \lambda y_n)\otimes y^*\|_\pi \\ &= \lim_{n\to\infty} \|y_n\otimes T^*(y^*) - \lambda y^*\|_\pi = \|T^*(y^*) - \lambda y^*\|. \end{aligned}$$

As above we conclude that $T = \lambda\,\mathrm{Id}_{\mathfrak{X}}$. □

Let us assume that $\mathfrak{X}$ has a bounded shrinking basis $\mathscr{X} = \{x_n\}_{n=1}^\infty$ whose associated sequence of coefficient functionals is $\mathscr{X}^* = \{x_n^*\}_{n=1}^\infty$. Then a basis $\{z_n\}_{n=1}^\infty$ of $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ is induced if we arrange all tensors $x_n\otimes x_m^*$ for $r, s \in \mathbb{N}$ in a right way. For, if $m \in \mathbb{N}$ let $n \in \mathbb{N}$ so that $(n-1)^2 < m \le n^2$ we write

$$\sigma(m) = \begin{cases} \left(m-(n-1)^2, n\right) & if \quad (n-1)^2+1 \le m \le (n-1)^2+n, \\ \left(n, n^2-m+1\right) & if \quad (n-1)^2+n \le m \le n^2. \end{cases}$$

Therefore $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ becomes a bijective function and it suffices to put $z_n = x_{\sigma_1(n)} \otimes x^*_{\sigma_2(n)}$ (cf. [9], [10]).

Theorem 7. *(cf.* [3]*) If* $\delta \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ *there are unique sequences* $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ *and* $\{\mathfrak{y}_u^v\}_{u,v\in\mathbb{N}}$ *so that if* $u, v \in \mathbb{N}$ *then*

$$\delta\left(z_{\sigma^{-1}(u,v)}\right) = (\mathfrak{h}_u - \mathfrak{h}_v)\, z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty}\left(\mathfrak{y}_u^n \cdot z_{\sigma^{-1}(n,v)} - \mathfrak{y}_n^v \cdot z_{\sigma^{-1}(u,n)}\right).$$

We say that $\mathfrak{h} = \mathfrak{h}[\delta]$ *and that* $\eta = \eta[\delta]$ *are the* $\mathfrak{h}$ *and* $\mathfrak{y}$ *sequences of* δ *respectively. Indeed,* $\mathfrak{h}[\delta] = \left\{\left\langle \delta(z_{n^2}), z_{n^2}^*\right\rangle\right\}_{n=1}^{\infty}$ *and* $\eta[\delta] = \left\{\left\langle \delta(z_{n^2}), z_{m^2}^*\right\rangle\right\}_{n,m=1}^{\infty}$. *An* $\mathscr{X}$*-Hadamard bounded derivation on* $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *is any derivation with null* η *sequence. In* [3] *it is proved that they constitute a complementary Banach subspace of* $\mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.

Problem 8. *Characterize the class of Hadamard derivations intrinsically or independently of any basis.*

Problem 9. *What is the relation between* $\mathscr{X}$*-Hadamard and* $\mathscr{B}$*-derivations?- We conjecture that any* $\mathscr{X}$*-Hadamard derivation is realized as a* $\mathscr{B}$*-derivation by a multiplier operator of both* $\mathfrak{X}$ *and* $\mathfrak{X}^*$ *relative to the basis* $\mathscr{X}$ *and* $\mathscr{X}^*$ *respectively. As a consequence of Lemma 6 the corresponding supporting operator must be unique up to a constant multiple of* $\mathrm{Id}_{\mathfrak{X}}$.

REFERENCES

[1] A. L. Barrenechea & C. C. Peña: *Some remarks about bounded derivations on the Hilbert algebra of square summable matrices*. Matematicki Vesnik, 57, No. 4, 78–95, (2005).

[2] A. L. Barrenechea & C. C. Peña: *On innerness of derivations on S(H)*. Lobachevskii J. of Math., Vol. 18, 21–32, (2005).

[3] A. L. Barrenechea & C. C. Peña: *On the structure of derivations on certain non-amenable nuclear Banach algebras.* Preprint, (2007).

[4] B. E. Johnson: *Norms of derivations on* $\mathscr{L}(\mathfrak{X})$. Pac. J. Math. 38, 465–469, (1971).

[5] B. E. Johnson: *Cohomology in Banach algebras.* Mem. Amer. Math. Soc., **127,** (1972).

[6] I. Kaplansky: *Modules over operator algebras.* Amer. J. Math., 75, 839–853, (1953).

[7] J. Kyle: *Norms of derivations.* J. London Math. Soc. (2), 16, 297–312, (1977).

[8] V. Runde: *Lectures on amenability.* Springer-Verlag, Berlin, Heidelberg, N. Y., (2002).

[9] R. Schatten: *A theory of cross spaces.* Ann. of Math. Studies **26**. Princeton University Press, (1950).

[10] I. Singer: *Bases in Banach spaces I.* Springer-Verlag, Berlin-Heidelberg-N. Y., (1970).

[11] J. G. Stampfli: *The norm of a derivation.* Pac. J. Math., 33, 737–747, (1970).

UNCPBA. DEPARTAMENTO DE MATEMÁTICAS-NUCOMPA.