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A REMARK ON AN APPROXIMATE FUNCTIONAL EQUATION FOR $\zeta(s)$

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ABSTRACT. We derive an approximate functional equation for Riemann zeta function in the critical strip with sharp error term using a combinatorial identity.

1. INTRODUCTION

Perhaps the simplest of all approximate formulas for $\zeta(s)$, the Riemann zeta function, is

$$\zeta(s) = \sum_{n \leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) ,$$

which holds uniformly for $0 < \sigma_0 \leqslant \sigma$, $|t| < \frac{2\pi x}{C}$, where C is a given constant greater than 1 (here, as usual, $s = \sigma + it$). See [3], pg. 77.

In the present note we present an approximate functional equation for $\zeta(s)$ in the critical strip (Theorem 2) which differs from the classical one and depends on a certain combinatorial identity (Lemma 2). Our approximate functional equation has a sharp error term but the main term is combinatorially complicated. We give some evidence that this main term behaves like a jump function.

No use of this functional equation is made in this note.

2. THE APPROXIMATE FUNCTIONAL EQUATION

Our main results are Theorems 1 and 2. Theorem 2 gives an approximate functional equation with sharp error term.

Theorem 1. *If $s = \sigma + it$ with $0 < \sigma < 1$ then*

$$\zeta(s) = \frac{\sin(\pi s)}{(1-2^{1-s})\pi} \int_0^\infty x^{-s} \left(\sum_{n=1}^{\infty} \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)} \left\{ \frac{3}{2}n + \frac{1}{2}x - \frac{1}{4} \right\} \right) dx. \quad (1)$$

For any real number x, let $[x]$ denote the integer part of x.

Theorem 2. *Assume $N = [c_1 t]$ with $\frac{\pi}{2\log 4} < c_1$. Then the following formula holds uniformly if $0 < \sigma \leqslant \sigma_0 < 1$, $t > 0$:*

$$(1-2^{1-s})\zeta(s) = \frac{1}{2} \sum_{j=0}^{2N-1} \frac{(-1)^j}{(1+j)^s} a_{j,N} + O(t^{-\sigma} e^{-(c_1 \log 4 - \frac{\pi}{2})t}),$$

where $a_{j,N} = \sum_{n=[\frac{j}{2}]+1}^{N} \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n - j - \frac{3}{2})$.

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3. Proofs

We need to recall

Lemma 3. *Let $Q(x)$ be a meromorphic function of x having no poles on the positive real axis and such that $x^aQ(x) \to 0$ both when $x \to 0$ and $x \to \infty$. Also $\int_{C_{\rho_i}} (-z)^{a-1}Q(z)dz \to 0$ if $i \to \infty$ where C_{ρ_i} is a sequence of circles (squares) centered at the origin with increasing radii (diameters) tending to infinity. Then*

$$\int_0^\infty x^{a-1}Q(x)dx = \frac{\pi}{\sin(\pi a)}\sum r\,, \tag{2}$$

where $\sum r$ denotes the sum of the residues of $(-z)^{a-1}Q(z)$, and the residues in $\sum r$ are added according to their distance to the origin. Here $(-z)^{a-1} = e^{(a-1)\log(-z)}$, where $-\pi \leqslant Arg(-z) < \pi$.

This lemma is well-known and we refer the reader to [4] pg. 117.

Proof of Theorem 1. We have

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)}\left\{\frac{3}{2}n+\frac{1}{2}x-\frac{1}{4}\right\}, \tag{3}$$

This formula is proved in Lemma 2 below.

Let a be a real number such that $0 < a < \frac{1}{2}$ and let $Q(x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)}$. It is not difficult to show that $\int_{C_{\rho_i}} (-z)^{a-1}Q(z)dz \to 0$ if $i \to \infty$, where C_{ρ_i} is the square centered at zero of side $2i+1$. Also $x^aQ(x) \to 0$ if $x \to 0$. Applying Lemma 1 we obtain

$$\int_0^\infty x^{a-1}\left(\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)}\right)dx = \frac{\pi}{\sin(\pi a)}\sum_{n=1}^\infty (-1)^{n-1}n^{a-1}. \tag{4}$$

Setting $1-a = s = \sigma + it$, using analytic continuation, formula (3) and $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$ $(\sigma > 0)$, we arrive to formula (1). □

Lemma 4. *The following identity holds*

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)}\left\{\frac{3}{2}n+\frac{1}{2}x-\frac{1}{4}\right\}.$$

Proof. First we have

$$\sum_{k=1}^K \frac{b_1\dots b_{k-1}}{x(x+a_1)\dots(x+a_k)}(x+a_k-b_k) = \frac{1}{x} - \frac{b_1\dots b_K}{x(x+a_1)\dots(x+a_K)},$$

which follows from writing the right hand side as $A_0 - A_K$ and noticing that each term on the left is $A_{k-1} - A_k$. Replace x by $(n+x)^2$, a_k by $-k^2$, b_k by $k(\frac{1}{2}-k)$ and $K = n-1$. Multiply everything by $(n+x)(-1)^{n-1}$ and add from $n=1$ to N. Then $b_1\dots b_{k-1} = (-1)^{k-1}\frac{(2k-2)!}{4^{k-1}}$ and

$$\sum_{n=1}^N \frac{(-1)^{n-1}}{(n+x)} - \sum_{n=1}^N \frac{(-1)^{n-1}b_1\dots b_{n-1}}{(2n-1+x)\dots(1+x)}$$
$$= \sum_{n=1}^N\sum_{k=1}^{n-1} \frac{(-1)^{n-1}}{(n+k+x)\dots(n-k+x)}b_1\dots b_{k-1}\left((n+x)^2-\frac{k}{2}\right). \tag{5}$$

We define

$$\varepsilon_{n,k}(x) := (-1)^{n+k}\frac{(2k-2)!(\frac{n+x}{2}+\frac{1}{4})}{4^{k-1}(n+k+x)\dots(n-k+1+x)}.$$

Then the last formula of (5) is equal to

$$\sum_{n=1}^{N}\sum_{k=1}^{n-1}(\varepsilon_{n,k}(x)-\varepsilon_{n-1,k}(x)) = \sum_{k=1}^{N}\varepsilon_{N,k}(x) - \sum_{k=1}^{N}\varepsilon_{k,k}(x).$$

Now notice that $\sum_{k=1}^{N}\varepsilon_{N,k}(x)\to 0$ if $N\to\infty$ and $0<x<1$. Thus by analytic continuation

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}b_1\dots b_{n-1}}{(2n-1+x)\dots(1+x)} - \sum_{k=1}^{\infty}\varepsilon_{k,k}(x),$$

which is (3). □

To prove Theorem 2 we need the following lemma.

Lemma 5. *If* $1\leqslant n$, $-1<\sigma<1$, $t>0$ *then*

(i)

$$\int_0^\infty \frac{x^{-s}}{(2n+x)(2n-1+x)\dots(1+x)}dx = \frac{\pi}{(2n-1)!\sin(\pi s)}\sum_{j=0}^{2n-1}\binom{2n-1}{j}(-1)^j(j+1)^{-s}.$$

(ii) *For the same values of n and t and with* $-1<\sigma\leqslant\sigma_0<1$ *there exists a constant* $c_0=c_0(\sigma_0)$ *(depending only on* σ_0*) such that the absolute value of the integral in (i) is bounded by*

$$\frac{c_0}{|\sin(\pi s)|}\frac{n^{1-\sigma}e^{t\pi/2}}{(2n)!}.$$

Proof. Apply Lemma 1 to the left-hand side of (i) to obtain

$$\frac{\pi}{\sin(\pi s)}\left(\frac{1^{-s}}{(2n-1)!0!} - \frac{2^{-s}}{(2n-2)!1!} + \dots - \frac{(2n)^{-s}}{(2n-1)!}\right),$$

which is the right-hand side of (i). (ii) is proved as follows. By Lemma 1 we have to evaluate $\int_\gamma := \int_\gamma \frac{(-z)^{-s}}{(2n+z)(2n-1+z)\dots(1+z)}dz$ where γ is a positively oriented curve enclosing $-1,-2,\dots,-2n$.

Take γ to be the rectangle with vertices $-\varepsilon+i2n$, $-2n-1+i2n$, $-2n-1-i2n$, $-\varepsilon-i2n$, $(0<\varepsilon<1)$. We parametrize $\gamma=\gamma(\tau)$, $\tau\in[0,cn]$ and $|\gamma'(\tau)|=1$ with c depending on ε but bounded for any ε. Therefore we have

$$|\int_\gamma|\leqslant\int_0^{cn}|\frac{(-\gamma(\tau))^{-s}}{(2n+\gamma(\tau))\dots(1+\gamma(\tau))}|.|\gamma'(\tau)|d\tau\leqslant$$

$$\max_{z\in\gamma}|\frac{1}{(2n+z)\dots(1+z)}|\int_0^{cn}|(-\gamma(\tau))^{-s}|d\tau,$$

and

$$\int_0^{cn}|(-\gamma(\tau))^{-s}|d\tau\leqslant\int_0^{cn}|\gamma(\tau)|^{-\sigma}e^{tArg(-\gamma(\tau))}d\tau\leqslant e^{\frac{t\pi}{2}}\int_0^{cn}|\gamma(\tau)|^{-\sigma}d\tau.$$

Now we claim that $\max_{z\in\gamma}|\frac{1}{(2n+z)\dots(1+z)}|\leqslant\frac{1}{(2n)!}+\delta(\varepsilon)$ with $\delta(\varepsilon)\to 0$ if $\varepsilon\to 0$. This would prove (ii). To prove the above inequality we observe that by symmetry it is enough to compute a bound on the segments $[-\varepsilon,-\varepsilon+i2n],[-2n-1+i2n,-\varepsilon+i2n]$. For the first segment, it is easily seen that for any point z on it we have that $|j+z| = dist(z,-j)$ increases if z moves upwards on the segment. Thus the maximum is obtained on $z=-\varepsilon$. On the second segment it can be seen that for any point z on it we have $2n \leqslant |z+j|$. This shows that there the maximum is less than $\frac{1}{(2n)^{2n}}$. This proves the claimed inequality. Letting ε tend to zero we get (ii). □

Proof of Theorem 2. Write, for short, $f(x,n) := \frac{1}{(2n+x)\dots(1+x)}(\frac{3n}{2}+\frac{x}{2}-\frac{1}{4})$. So (1) is written, interchanging summation and integration, as

$$\zeta(s) = \frac{\sin(\pi s)}{(1-2^{1-s})\pi}\sum_{n=1}^{N}\int_0^\infty \frac{(2n-2)!x^{-s}f(x,n)}{4^{n-1}}dx + \frac{\sin(\pi s)}{(1-2^{1-s})\pi}\sum_{n=N+1}^{\infty}\int_0^\infty \frac{(2n-2)!x^{-s}f(x,n)}{4^{n-1}}dx, \quad (6)$$

Let $N=[c_1 t]$ with $\frac{\pi}{2\log 4} < c_1$. For $n \geqslant N+1$ the last sum of (6) can be estimated using ii) of Lemma 3:

$$\frac{|\sin(\pi s)|}{|(1-2^{1-s})|\pi}\sum_{n=N+1}^{\infty}\Big|\int_0^\infty \frac{(2n-2)!x^{-s}f(x,n)}{4^{n-1}}dx\Big| \leqslant$$

$$\frac{c_0}{|(1-2^{1-s})|\pi}\sum_{n=N+1}^{\infty}\frac{(2n-2)!}{4^{n-1}}\Big(\frac{3n}{2}\frac{n^{1-\sigma}e^{\frac{t\pi}{2}}}{2n!}+\frac{1}{2}\frac{n^{1-(\sigma-1)}e^{\frac{t\pi}{2}}}{2n!}+\frac{1}{4}\frac{n^{1-\sigma}e^{\frac{t\pi}{2}}}{2n!}\Big)$$

$$\leqslant \frac{9c_0 e^{t\pi/2}}{|1-2^{1-s}|8\pi}\sum_{n=N+1}^{\infty}\frac{n^{1-\sigma}}{(2n-1)4^{n-1}} \leqslant O\Big(\frac{e^{t\pi/2}N^{-\sigma}}{4^N}\Big) = O\big(t^{-\sigma}e^{-(c_1\log 4-\frac{\pi}{2})t}\big)$$

For the first sum in (6) we use (i) of Lemma 3. □

Remark 1. From Theorem 1 one has the following curious formula:

Corollary 1. *If* $f(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{ArcTanh}(\frac{1-e^{-x}}{2})e^{-x}x^{s-1}dx$ *then*

$$-\zeta(s)\frac{(1-2^{1-s})}{2} = f(s)-f(s-1)-\frac{1}{2}\Big(\frac{1}{1^s}+\frac{1}{3.2^s}+\frac{1}{3^2.3^s}+\frac{1}{3^3.4^s}+\frac{1}{3^4.5^s}\dots\Big)$$

Hint. Recall that $\int_0^\infty e^{-jx}x^{s-1}dx = \Gamma(s)j^{-s}$ for suitable s and j. Thus Lemma 3 (i) is equal to

$$\frac{\pi}{(2n-1)!\sin(\pi s)\Gamma(s)}\int_0^\infty (1-e^{-x})^{2n-1}e^{-x}x^{s-1}dx.$$

Using this in formula (1), interchanging summation and integration and using that

$$\sum_{n=1}^{\infty}\frac{\frac{3n}{2}-\frac{1}{4}}{4^{n-1}(2n-1)}\alpha^{2n-1} = \mathrm{ArcTanh}(\frac{\alpha}{2})-\frac{3\alpha}{\alpha^2-4},$$

and

$$\sum_{n=1}^{\infty}\frac{\alpha^{2n-1}}{4^{n-1}(2n-1)} = 2\,\mathrm{ArcTanh}(\frac{\alpha}{2}),$$

we get

$$\zeta(s) = \frac{f(s)-f(s-1)}{(1-2^{1-s})}-\frac{3}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{(1-e^{-x})e^{-x}x^{s-1}}{(1-e^{-x})^2-4}dx.$$

This formula proves the corollary after some simplifications. □

4. On the coefficient $a_{j,N}$

Recall the definition of $a_{j,N}$, the coefficient of $\frac{(-1)^j}{(1+j)^s}$ in Theorem 2:

$$a_{j,N} = \sum_{n=[j/2]+1}^{N} \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n-j-3/2).$$

The author noticed numerically that this coefficient behaved like a jump function. More precisely

$$a_{j,N} \approx \begin{cases} 2 & \text{if } 0 \leqslant j \leqslant N-2 \\ 1 & \text{if } j = N-1 \\ 0 & \text{if } N \leqslant j \leqslant 2N-1. \end{cases}$$

Here we give some evidence of this fact. An unknown referee has kindly provided part of the proof below.

As $a_{0,N} = 2(1-4^{-N})$ we assume that $1 \leqslant j$. Also if $n \leqslant [j/2]$ then $\binom{2n-1}{j} = 0$ since $j > 2n-1$. Thus we write for $1 \leqslant j$

$$a_{j,N} = 6\sum_{n=1}^{N} \frac{1}{4^n}\binom{2n-1}{j} - 4\sum_{n=1}^{N} \frac{1}{4^n}\binom{2n-2}{j-1},$$

where we have used that $\binom{2n-1}{j} = \frac{2n-1}{j}\binom{2n-2}{j-1}$ as long as $1 \leqslant j$.

Now we will show that

$$2 = a_{j,N} + \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n-j-3/2) = a_{j,N} + \mathrm{Tail}(N,j). \tag{7}$$

The tail will be seen to be small in a sense explained below. But the middle formula of (7) can be written as (curves are oriented in the usual way)

$$\begin{aligned} 6\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n-1}{j} - 4\sum_{n=1}^{\infty} \frac{1}{4^n}\binom{2n-2}{j-1} \\ = \frac{1}{2\pi i}\Big(6\sum_{n=1}^{\infty} \int_{|z|=1/2} \frac{(1+z)^{2n-1}}{4^n z^{j+1}} dz - 4\sum_{n=1}^{\infty} \int_{|z|=1/2} \frac{(1+z)^{2n-2}}{4^n z^j} dz\Big) \\ = \frac{1}{2\pi i}\int_{|z|=1/2} \frac{1}{4z^j(1-\frac{(1+z)^2}{4})}\Big(\frac{6(1+z)}{z} - 4\Big) dz. \end{aligned}$$

Now one deforms the curve $|z| = 1/2$ to $|z| = r$, computing the residues at $z = 1, -3$ ($-2, 0$ respectively). Notice that the integral over the curve $|z| = r$ tends to zero if $r \to \infty$. This proves (7).

The evidence that our function $a_{j,N}$ behaves like a jump function is given by:

a) $\mathrm{Tail}(N,j) = O(1)$. For any fixed $0 < \delta < 1$, $\mathrm{Tail}(N,j) \to 0$ uniformly in j if $1 \leqslant j < N\delta$ and $N \to \infty$.
b) For any fixed $0 < \delta < 1$, $\mathrm{Tail}(N,j) \to 2$ uniformly in j if $N.(1+\delta) < j \leqslant 2N$ and $N \to \infty$.

Proof. As above the tail can be written using residues as

$$\begin{aligned}&\mathrm{Tail}(N,j)\\&=6.\frac{1}{2\pi i}\int_{|z|=1/2}\frac{(1+z)^{2N+1}}{4^N z^{j+1}(4-(1+z)^2)}dz-4.\frac{1}{2\pi i}\int_{|z|=1/2}\frac{(1+z)^{2N}}{4^N z^{j}(4-(1+z)^2)}dz\\&=6.S_1(N,j)-4.S_2(N,j).\end{aligned}$$

We will compare the above integrals with the more suitable

$$S_1'(N,j)=\frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{1}{4(1-z)}.\frac{(1+z)^{2N+1}}{4^N z^{j+1}}dz,$$

$$S_2'(N,j)=\frac{1}{2\pi i}\int_{|z|=\frac{1}{2}}\frac{1}{4(1-z)}.\frac{(1+z)^{2N}}{4^N z^{j}}dz.$$

Indeed we will see that uniformly in $0\leqslant j\leqslant 2N$, $i=1,2$ one has

$$S_i(N,j)-S_i'(N,j)=o(1)\ as\ N\to\infty. \tag{8}$$

Also, we will prove for fixed $0<\delta<1$, $i=1,2$

$$S_i'(N,j)\to 0 \text{ uniformly in } j \text{ if } 1\leqslant j\leqslant N\delta; N\to\infty, \tag{9}$$

$$S_1'(N,j)+S_1'(N,2N-j)=\frac{1}{2};\ S_2'(N,j)+S_2'(N,2N-j+1)=\frac{1}{4} \tag{10}$$

Observe that a), b) follows from (8), (9), (10) and the fact that $0\leqslant S_i'(N,j)$.

To prove (8) say, for $i=1$, notice that

$$S_1(N,j)-S_1'(N,j)=\frac{1}{2\pi i}\int_{\gamma_1+\gamma_2}g(z)\frac{(1+z)^{2N+1}}{4^N z^{j+1}}dz$$

where $g(z)$ is a regular function on $|z|=1$; γ_1 is the curve given by $\{|z|=1,\varepsilon(N)\leqslant Arg(z)\leqslant 2\pi-\varepsilon(N)\}$ and $0<\varepsilon(N)$ is chosen so that $|1+z|\leqslant 2(1-\frac{\mathrm{Log}N}{2N})$ on γ_1. Now it is not difficult to see that $\varepsilon(N)$ tends to zero if $N\to\infty$. Also we denote γ_2 the curve $\{|z|=1, -\varepsilon(N)\leqslant Arg(z)\leqslant\varepsilon(N)\}$; so that the length of γ_2 tends to zero as $N\to\infty$.

Now on γ_1 the above integral is by the maximum modulus principle

$$\int_{\gamma_1}=O(\frac{(1-\frac{\mathrm{Log}N}{2N})^{2N}}{1})=o(1)$$

Now the integral over γ_2 tends to zero because the length of γ_2 tends to zero. This proves $S_1(N,j)-S_1'(N,j)=o(1)$ as $N\to\infty$. The proof for $i=2$ is similar.

Now in the definition of $S_1'(N,j)$ deforming the curve $|z|=1/2$ to a curve $|z|=2$ and computing the residue at $z=1$ one has

$$S_1'(N,j)=\frac{1}{2}+\frac{1}{2\pi i}\int_{|z|=2}\frac{1}{4(1-z)}.\frac{(1+z)^{2N+1}}{4^N z^{j+1}}dz$$

This last integral is $-S_1'(N,2N-j)$ making the change of variable $z=1/w$. This proves (10) (case $i=2$ is similar).

Finally to prove (9) notice that

$$|S_1'(N,j)|=|\frac{1}{2\pi i}\int_{|z|=r<1}\frac{1}{4(1-z)}.\frac{(1+z)^{2N+1}}{4^N z^{j+1}}|<<_r(\frac{(1+r)^2}{4})^N\frac{1}{r^j}\leqslant(\frac{(1+r)^2}{4.r^{\delta}})^N$$

But $(\frac{(1+r)^2}{4.r^{\delta}})<1$ for $r=\frac{\delta}{2-\delta}$. Again case $i=2$ is similar.

□

Notice that $S_1'(N,N) = \frac{1}{4}$. This follows from (10) with $j = N$.

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