

## UNIVERSAL QUASIVARIETIES OF ALGEBRAS

M.E. ADAMS AND W. DZIOBIAK

ABSTRACT. Two different notions of universal, one due to Hedrlín and Pultr in the 1960s and the other due to Sapir in the 1980s, are discussed, as well as the relationship between them. Some of the historical perspective and mathematical motivation lying behind them is also included, together with a brief overview of a variety meriting further investigation in this context.

### 1. INTRODUCTION

In the 1960s, Hedrlín and Pultr introduced the notion of a universal quasivariety. In §2, universal quasivarieties are discussed, as well as some of their motivation: illustrating examples include unary algebras, lattices, and graphs.

In the 1970s, pseudocomplemented distributive lattices became the subject of intense investigation following Lee's elegant description of the lattice of subvarieties in terms of their equational bases. Research on pseudocomplemented distributive lattices is closely linked to the development of the two notions of universal discussed here, and will be addressed in §3.

In the 1980s, Sapir introduced the notion of a  $Q$ -universal quasivariety. In §4,  $Q$ -universal quasivarieties are discussed, as well as some of the underlying motivation. Illustrating examples include semigroups, as well as the  $Q$ -universality of unary algebras, lattices, and graphs.

Bringing us up to the 2000s, connections between the two notions are considered in §5.

This is an active area of research and in §6, via monadic Boolean algebras, we seek to suggest some pertinent open problems that require minimal technical background.

Our primary objective is to provide an introduction to the topic in question. As such, we make no claim to a comprehensive treatment, nor will we necessarily attempt to state results in their full generality.

---

2000 *Mathematics Subject Classification*. Primary: 08C15, 08A35.

*Key words and phrases*. Quasivariety, variety, universal,  $Q$ -universal, unary algebra, lattice, pseudocomplemented distributive lattice, semigroup, monadic Boolean algebra, graph.

This is an expanded version of a presentation given at IX Congreso Dr. Antonio Monteiro, Centenario del Nacimiento de Antonio Monteiro, Bahía Blanca, 30 de mayo al 1ro de junio de 2007.

Grant acknowledgment: While working on this paper, the authors were supported by US CRDF grant KYM1 - 2852 - BI - 07.

## 2. UNIVERSAL IN THE SENSE OF HEDRLÍN AND PULTR

For an algebra  $A$ , the automorphisms of  $A$  under composition form a group, denoted  $\text{Aut}(A)$ , begging the question of when, for a given group  $K$ , there exists an algebra  $A$  such that  $\text{Aut}(A) \cong K$ . As Cayley's classical theorem states, for a group  $K$ , there always exists a group  $G$  for which  $\text{Aut}(G) \cong K$ . Other classes of algebra also have this property; for example, Birkhoff [26] showed that there always exists a lattice  $L$  for which  $\text{Aut}(L) \cong K$  (in fact,  $L$  may be chosen to be a distributive lattice). Neither is there any need to limit oneself to algebras; as shown by Frucht [28], for any group  $K$ , there exists a graph  $G$  for which  $\text{Aut}(G) \cong K$ , where  $\text{Aut}(G)$  denotes the compatible bijections of  $G$  to itself closed under composition. Taking the graph  $G = (V, E)$ , Frucht [28] went on to give an alternative proof of Birkhoff's result that there exists a lattice  $L$  for which  $\text{Aut}(L) \cong K$  (where  $L$  is taken to be the lattice whose members are those of  $V$  and  $E$  together with a least member  $0$  and a maximum element  $1$ , and for which  $v < e$  iff  $v \in e$  with  $v \in V$  and  $e \in E$ ). Although providing a simple and visual proof of Birkhoff's theorem, lattices constructed in this manner are rarely distributive.

More generally, the endomorphisms of an algebra  $A$  under composition form a *monoid* (that is, a semigroup with identity), denoted  $\text{End}(A)$ . Analogously, as shown by Armbrust and Schmidt [16], for a monoid  $M$ , there exists an algebra  $A$  such that  $\text{End}(A) \cong M$ , which, as shown by Hedrlín and Pultr [41], may be chosen to be an algebra of type  $(1, 1)$  (that is, a unary algebra with two operations). In the early sixties a number of mathematicians (see Grätzer [34], page 68) were considering the question of when, for a given monoid  $M$ , there exists an algebra  $A$  for which  $\text{End}(A) \cong M$ . However, it was Hedrlín and Pultr who raised the bar, as we shall see momentarily.

A class of algebras of the same fixed type that is closed under isomorphisms, subalgebras, direct products, and ultraproducts is called a *quasivariety*. If it is also closed under homomorphic images, then it is a *variety* —every variety is a quasivariety, but not vice versa. The quasivarieties contained in a quasivariety  $\mathbf{K}$  form a lattice  $L(\mathbf{K})$  under inclusion, as do the varieties contained in a variety denoted  $L_V(\mathbf{K})$ .

Following Pultr [60], Hedrlín and Pultr [41], and Vopěnka, Hedrlín and Pultr [75], a quasivariety  $\mathbf{K}$  is *universal* if every category of algebras of finite type is isomorphic to a full subcategory of  $\mathbf{K}$ . Equivalently, the category  $\mathbf{G}$  of all graphs together with all compatible mappings is isomorphic to a full subcategory of  $\mathbf{K}$ . Since, as shown by them, for any monoid  $M$  there exists a graph  $G$  such that  $\text{End}(G) \cong M$ , it immediately follows that if  $\mathbf{K}$  is universal, there is also an algebra  $A$  in  $\mathbf{K}$  such that  $\text{End}(A) \cong M$ .

Typically, to establish that a quasivariety  $\mathbf{K}$  is universal, a suitable functor  $\Phi$  from the category  $\mathbf{G}$  of all graphs together with all compatible mappings, or, equivalently, from the category of all directed graphs, to  $\mathbf{K}$  is presented, which is then used to show that  $\mathbf{G}$  is isomorphic to a full subcategory of  $\mathbf{K}$ . Constructions of this type are known as *šip-constructions*, a lucid exposition of which may be found, for example, in Mendelsohn [58]. Such functors were given by Hedrlín and Pultr in their pioneering work establishing, inter alia, that the variety  $\mathbf{U}_n$  of all unary algebras with  $n$  unary operations is universal iff  $n \geq 2$  (see also Pultr and Sichler [61] and Sichler [68], [70], [71]). If, in addition, a functor  $\Phi$  can

be chosen so that finite graphs are sent to finite algebras, then  $\mathbf{K}$  is said to be *finite-to-finite universal*. In fact, for  $n \geq 2$ ,  $\mathbf{U}_n$  is finite-to-finite universal, a point to which we shall refer later.

If the underlying motivation behind the introduction of universal quasivarieties is the representation of monoids as endomorphism monoids of algebras, then, as shown by Hedrlín and Sichler [42], it is at its sharpest for finite-to-finite universal quasivarieties: for a finite-to-finite universal quasivariety  $\mathbf{K}$  and monoid  $M$ , if  $\kappa \geq |M|$  is infinite, then there exists a family  $(A_i \in \mathbf{K} : i < 2^\kappa)$  such that, for  $i, j < 2^\kappa$ ,  $\text{End}(A_i) \cong M$ ,  $|A_i| = \kappa$ , and there are no homomorphisms from  $A_i$  to  $A_j$  whenever  $i \neq j$ ; if  $M$  is finite, then there exists a countably infinite family  $(A_i \in \mathbf{K} : i < \omega)$  of finite algebras such that, for  $i, j < \omega$ ,  $\text{End}(A_i) \cong M$  and there are no homomorphisms from  $A_i$  to  $A_j$  whenever  $i \neq j$ .

Amongst the earliest varieties of algebras shown to be universal was the variety of bounded lattices, Grätzer and Sichler [39] (later shown to be finite-to-finite universal, Adams and Sichler [11]). In doing so, they gave a functor from the category of *triangle-connected graphs* to the variety of bounded lattices, thereby making use of the fact that the category of triangle-connected graphs is universal, Hell [43]. Employing a category of graphs with special properties is no longer so unusual and, over time, many categories of graphs with special properties have been shown to be (finite-to-finite) universal—see Hell and Nešetřil [44]—as have many (quasi)varieties of algebras—see Pultr and Trnková [62] and, for some more recent references, Adams, Adaricheva, Dziobiak, and Kravchenko [2].

### 3. AN INFLUENTIAL VARIETY: PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

A *pseudocomplemented distributive lattice*  $(L, \vee, \wedge, *, 0, 1)$  is a bounded distributive lattice  $(L, \vee, \wedge, 0, 1)$  where, for  $x, y \in L$ ,  $x \wedge y = 0$  iff  $y \leq x^*$ . Pseudocomplemented distributive lattices are a generalization of Boolean algebras which, as shown by Ribenboim [63], form an equational class—equivalently, by Birkhoff's classical theorem, a variety  $\mathbf{B}$ . Since pseudocomplemented distributive lattices have played a rôle greater than that of an illustrative example, a brief overview is appropriate.

In 1970, Lee [55] showed that  $L_V(\mathbf{B})$  is an  $\omega + 1$ -chain

$$\mathbf{B}_{-1} \subset \mathbf{B}_0 \subset \mathbf{B}_1 \subset \dots \subset \mathbf{B}_n \subset \dots \subset \mathbf{B},$$

where  $\mathbf{B}_{-1}$ ,  $\mathbf{B}_0$ , and  $\mathbf{B}_1$  are the varieties of one-element algebras, Boolean algebras, and Stone algebras, respectively. Moreover, Lee showed that, for  $1 \leq n < \omega$ ,  $L \in \mathbf{B}_n$  iff the equation

$$(x_1 \wedge \dots \wedge x_n)^* \vee \bigvee_{1 \leq i \leq n} (x_1 \wedge \dots \wedge x_i^* \wedge \dots \wedge x_n)^* = 1$$

holds in  $L$ .

Following quickly on the heels of Lee's elegant paper were another three significant papers, Lakser [54] and Grätzer and Lakser [36], [37], where a number of properties of

pseudocomplemented distributive lattices were considered in detail —subdirect irreducibility, congruence extension, the standard semigroup of operators, amalgamation, injectivity. At the time, Grätzer and Lakser posed a number of interesting problems. Verifying one of their conjectures (see Grätzer [35]), Adams [1] and Wroński [76] independently showed that there are  $2^\omega$  quasivarieties of pseudocomplemented distributive lattices, that is  $|L(\mathbf{B}_\omega)| = 2^\omega$ . This was rapidly superseded by Grätzer, Lakser, and Quackenbush [38], showing that  $|L(\mathbf{B}_3)| = 2^\omega$  —since every quasivariety contained in  $\mathbf{B}_2$  is a variety, this is sharp. They also showed that  $L(\mathbf{B}_3)$  is non-modular —it already being known that, for any quasivariety  $\mathbf{K}$  of algebras, if  $L(\mathbf{K})$  is a modular lattice then it is distributive, Gorbunov [30]. Subsequently, Dziobiak [23] and Tropin [73] independently showed that  $L(\mathbf{B}_3)$  fails to satisfy any non-trivial lattice identity. In either case, they showed that the existence of a family of algebras with certain properties was sufficient to show that any non-trivial lattice identity would fail to hold in  $L(\mathbf{B}_3)$ . Since this is a point to which we will return, we will explicitly state Dziobiak’s criteria P1–(P4) as given in [23].

Interpreting the class-operators  $S, P$  in the inclusive sense, so that for example,  $S(\mathbf{K})$  denotes the class of all algebras isomorphic to a subalgebra of some algebra in  $\mathbf{K}$ , a quasivariety is a class  $\mathbf{K}$  of algebras of the same signature such that  $S(\mathbf{K}) = P(\mathbf{K}) = P_u(\mathbf{K}) = \mathbf{K}$  where  $P_u(\mathbf{K})$  denotes the class of all algebras isomorphic to an ultraproduct of algebras from  $\mathbf{K}$ .

Let  $N$  be a fixed infinite but countable set and  $\mathcal{P}_f(N)$  denote the set of all finite subsets of it. Suppose  $\mathbf{K}$  is a quasivariety of algebras of finite type that contains a family  $(A_X : X \in \mathcal{P}_f(N))$  of finite members satisfying the following conditions:

- (P1)  $A_\emptyset$  is a trivial member of  $\mathbf{K}$ ;
- (P2) if  $Z = X \cup Y$ , then  $A_Z \in SP(\{A_X, A_Y\})$ ;
- (P3) if  $X \neq \emptyset$  and  $A_X \in SP(\{A_Y\})$ , then  $X = Y$ ;
- (P4) if  $A_X$  is a subsystem of  $B \times C$  for finite  $B$  and  $C \in SP(\{A_W : W \in \mathcal{P}_f(N)\})$ , then there exists  $Y$  and  $Z$  with  $A_Y \in SP(B)$ ,  $A_Z \in SP(C)$ , and  $X = Y \cup Z$ .

Then  $L(\mathbf{K})$  fails to satisfy any non-trivial lattice identity. In particular, as shown in [23],  $\mathbf{B}_3$  contains such a family. (As does the variety of lattices, Dziobiak [24] —in point of fact, the variety of modular lattices  $\mathbf{M}_{3,3}$ .)

All to the well and good, but what, if anything, have pseudocomplemented distributive lattices to do with representing monoids as endomorphism monoids of algebras?

Recall that  $\mathbf{B}_{-1}$  is the variety of one-element algebras and  $\mathbf{B}_0$  the variety of Boolean algebras. Independently, Magill [56] and Schein [66] showed that Boolean algebras are recoverable from their endomorphism monoids: for Boolean algebras  $B_0, B_1$ , if  $End(B_0) \cong End(B_1)$ , then  $B_0 \cong B_1$ . In Adams, Koubek, and Sichler [10] it was shown that a similar result holds for the variety of Stone algebras  $\mathbf{B}_1$ , and, though there exist  $L_0, L_1 \in \mathbf{B}_2$  for which  $End(L_0) \cong End(L_1)$  with  $L_0 \not\cong L_1$ , it is a fact that, for  $L, L_0, L_1 \in \mathbf{B}_2$ , if  $End(L) \cong End(L_0) \cong End(L_1)$  and  $L_0 \not\cong L_1$ , then either  $L \cong L_0$  or  $L \cong L_1$ . Any hope that a pattern is being set whereby pseudocomplemented distributive lattices in  $\mathbf{B}_n$  are recoverable up to isomorphism as one of  $n$  from isomorphic endomorphism monoids is rapidly dashed since,

as shown in Adams, Koubek, and Sichler [9], there is a proper class of non-isomorphic algebras in  $\mathbf{B}_3$  each of which has a finite endomorphism monoid. Does this mean that  $\mathbf{B}_3$  is universal? Well, not exactly.

If  $I$  is a minimal prime ideal of a pseudocomplemented distributive lattice  $L$ , then  $\phi : L \rightarrow L$ , given by  $\phi(x) = 0$  if  $x \in I$  and 1 otherwise, is an endomorphism of  $L$ . Whenever  $L$  is non-trivial it has a minimal prime ideal, whereby  $|End(L)| \geq 2$ . Since, for any monoid  $M$ , including the one-element monoid, in a universal variety there exists a proper class of non-isomorphic algebras each of which has an endomorphism monoid isomorphic to  $M$ , it follows that  $\mathbf{B}_\omega$  is not universal. Nevertheless, more can be said.

Recall that Grätzer and Sichler [39] showed that the variety of bounded lattices is universal, that is the variety of lattices  $(L, \vee, \wedge, 0, 1)$  of type  $(2, 2, 0, 0)$ . The same cannot be said of the variety of lattices  $(L, \vee, \wedge)$  of type  $(2, 2)$ , since, for  $c \in L$ , the constant map  $\phi(x) = c$  for all  $x \in L$  is an endomorphism of  $L$  and, in particular,  $|End(L)| \geq |L|$ . Once again,  $|End(L)| \geq 2$  whenever  $L$  non-trivial. For lattices, the constant maps are recognizable as the left zeros of the endomorphism monoid, which, as noted above, always abound. However, as shown by Sichler, their existence clouds the underlying reality. In [69], Sichler showed that the category of all graphs is isomorphic to a subcategory of the variety of lattices whose morphisms are precisely all non-constant homomorphisms (that is all homomorphisms the image of which is not a singleton). It follows immediately, for example, that given any monoid  $M$ , there exists a proper class of non-isomorphic lattices such that, for each member  $L$ ,  $End(L)$  is isomorphic to a monoid  $M'$  which is a copy of  $M$  together with a set of left zeros (in fact,  $|L|$  many left zeros).

If, for some quasivariety  $\mathbf{K}$ , the category of all graphs is isomorphic to a subcategory of  $\mathbf{K}$  whose morphisms are precisely all non-constant homomorphisms, then  $\mathbf{K}$  is said to be *almost universal* and, as before, if there is a functor which sends finite graphs to finite algebras, then  $\mathbf{K}$  is said to be *finite-to-finite almost universal*. This is a notion to which we will return later.

Returning to pseudocomplemented distributive lattices, the non-trivial endomorphisms associated with minimal prime ideals have as an image  $\{0, 1\}$ , the constants. In this context, one might define a constant map to be one whose image is  $\{0, 1\}$ . As shown in [10], the category of all graphs is isomorphic to a subcategory of  $\mathbf{B}_3$  whose morphisms consist of precisely all non-constant homomorphisms of this type. As a consequence [9], for any monoid  $M$ , there is a proper class of non-isomorphic pseudocomplemented distributive lattices in  $\mathbf{B}_4$  such that, for each member  $L$ ,  $End(L)$  is isomorphic to a monoid  $M'$  which is a copy of  $M$  together with a finite set of left zeros —an analogous, but not so cleanly stated, property holds in  $\mathbf{B}_3$ .

It was considerations such as these, that led Demlová and Koubek to define a quasivariety  $\mathbf{K}$  with a subquasivariety  $\mathbf{M}$  as  *$\mathbf{M}$ -relatively universal* providing the category of all graphs is isomorphic to a subcategory of  $\mathbf{K}$  whose morphisms consist of all homomorphism which do not have an image in  $\mathbf{M}$  and as *relatively universal* providing there exists a subquasivariety  $\mathbf{M}$  for which  $\mathbf{K}$  is  *$\mathbf{M}$ -relatively universal*. As before, should there exist an appropriate

functor which sends finite graphs to finite algebras, then  $\mathbf{K}$  will be said to be *finite-to-finite  $\mathbf{M}$ -relatively universal* or *finite-to-finite relatively universal*, respectively. In this terminology, both the variety of lattices and  $\mathbf{B}_3$  are finite-to-finite relatively universal: the variety of lattices is finite-to-finite  $\mathbf{T}$ -universal for the trivial variety  $\mathbf{T}$  and  $\mathbf{B}_3$  is finite-to-finite  $\mathbf{B}_0$ -universal.

#### 4. UNIVERSAL IN THE SENSE OF SAPIR

Lattices that are isomorphic to  $L(\mathbf{K})$  for some quasivariety  $\mathbf{K}$  are called  *$Q$ -lattices* and, in particular, any lattice of the form  $L(\mathbf{K})$  is called the  *$Q$ -lattice* of  $\mathbf{K}$ . A long standing open problem asks for a characterization of  $Q$ -lattices —known as the Birkhoff-Maltsev problem.

Solutions to the problem have been found within certain classes of lattices —for example, Gorbunov and Tumanov [33] (Boolean lattices), Adaricheva and Gorbunov [13] (lattices of convex subsets of partially ordered sets), or Semenova [67] (lattices of partial suborders of partially ordered sets), see [2].

However, despite many known properties of  $Q$ -lattices, the problem continues to fight back. For example, as shown independently by Gorbunov [30] and Igošin [45], every  $Q$ -lattice satisfies  $SD_{\vee}$  (that is, every  $Q$ -lattice is *join-semidistributive*). As a matter of fact, Gorbunov and Tumanov [33] (see also Adaricheva, Gorbunov, and Tumanov [14]) showed that the least quasivariety of lattices that contains all  $Q$ -lattices coincides with the class of all  $SD_{\vee}$ -lattices. Encouraging a ray of hope, Tumanov [74] showed that every finite distributive lattice is a  $Q$ -lattice. However, Dziobiak [25] showed that if an element of a finite  $Q$ -lattice is a join of  $k$  atoms, then it contains at most  $2^k - 1$  atoms, thereby giving an example of a finite  $SD_{\vee}$ -lattice which is not a  $Q$ -lattice.

Building on [25], Adaricheva and Gorbunov [13] went on to define the notion of an *equa-closure operator* which ultimately proved crucial in showing that an atomistic and algebraic lattice is a  $Q$ -lattice if and only if it is isomorphic to the lattice of subsets of some algebraic lattice  $A$  which are closed under arbitrary lattice meets in  $A$  and under arbitrary lattice joins of non-empty chains in  $A$ , see Adaricheva, Dziobiak, and Gorbunov [12]. Recently, Adaricheva and Nation [15] have introduced the closely related notion of an *equa-interior operator*, in the process of which they have found an example of a finite  $Q$ -lattice which is not lower bounded in the sense of McKenzie —although it was known that not every finite lower bounded lattice is a  $Q$ -lattice [25], the converse had been an open problem since the early nineties.

As the reader may have come to suspect,  $Q$ -lattices are quite sophisticated in nature. Perhaps their complexity is best illustrated by the following notion due to Sapir [65].

A quasivariety  $\mathbf{K}$  of algebras of finite type is  *$Q$ -universal* providing that, for any quasivariety  $\mathbf{M}$  of finite type,  $L(\mathbf{M})$  is a homomorphic image of a sublattice of  $L(\mathbf{K})$ . This notion was introduced by Sapir in [65], where amongst other results it was shown that the variety of commutative 3-nilpotent semigroups is  $Q$ -universal. For any  $Q$ -universal quasivariety  $\mathbf{K}$ , noteworthy properties include  $|L(\mathbf{K})| = 2^{\omega}$ , as well as that  $L(\mathbf{K})$  contains a copy of a

free lattice on countably many generators and, hence, fails to satisfy any non-trivial lattice identity.

Many (quasi)varieties of algebras are now known to be  $Q$ -universal —see, for example, Gorbunov [32]. In particular, as shown by Gorbunov [31] (see also Kravchenko [52]), the variety  $\mathbf{U}_n$  of all unary algebras with  $n$  unary operations is  $Q$ -universal for every  $n \geq 1$  —thereby demonstrating that a  $Q$ -universal variety need not be universal. However, for undirected graphs, the two notions do coincide —as shown by Kravchenko [53], a quasivariety of undirected graphs is  $Q$ -universal if and only if it is (finite-to-finite) universal (which, in turn, is equivalent to the presence of a non-bipartite graph). Nonetheless, quasivarieties of directed graphs behave differently —see Kravchenko [51] and Sizyř [72], as well as Problem 21 of [2].

Sapir’s original approach to establishing that a quasivariety is  $Q$ -universal makes essential use of the notion of a *split system* as introduced by him in his Ph.D. Thesis [64]. Additionally there are now two other approaches Adams and Dziobiak [3] and Gorbunov [31]. In [3], it was shown that whenever a quasivariety  $\mathbf{K}$  contains a family  $(A_X : X \in \mathcal{P}_f(N))$  of finite members satisfying (P1)–(P4), then  $L(\mathbf{K})$  contains as a sublattice a copy of the ideal lattice  $I(F)$  of a free lattice on countably many generators, by virtue of which  $\mathbf{K}$  is  $Q$ -universal. In particular, it follows immediately from the above that both the variety of all lattices and the variety of pseudocomplemented distributive lattices are  $Q$ -universal, since  $\mathbf{M}_{3,3}$  and  $\mathbf{B}_3$  contain such families. The conditions of [31] are also known to guarantee that  $L(\mathbf{K})$  contains a copy of  $I(F)$ , whilst inspection reveals that each of the  $Q$ -universal quasivarieties established using the conditions of [65] do too. Whether  $L(\mathbf{K})$  containing a copy of  $I(F)$  is a necessary condition for  $\mathbf{K}$  to be  $Q$ -universal is not known —see Problems 14, 15, and 16 of [2].

## 5. A CONNECTION

Since the variety  $\mathbf{U}_1$  of all unary algebras with one unary operation is  $Q$ -universal but not universal, not every  $Q$ -universal quasivariety is universal. However, in [5], Adams and Dziobiak showed that every finite-to-finite universal quasivariety contains a family  $(A_X : X \in \mathcal{P}_f(N))$  of finite members satisfying (P1)–(P4). In other words, every finite-to-finite universal quasivariety is  $Q$ -universal.

An immediate application is to varieties of bounded lattices, a motivating example that led to [5]. In their outstanding paper [29], Goralčřk, Koubek, and Sichler characterized finite-to-finite universal varieties of bounded lattices as those containing a finite non-distributive simple lattice (see also McKenzie [57]). Since the variety generated by the five element non-distributive modular lattice is such a variety, it follows that there are  $2^\omega$   $Q$ -universal varieties of bounded lattices. That the two notions do not coincide for bounded lattices is verified in Adams and Dziobiak [6], where it is shown that there are  $2^\omega$   $Q$ -universal varieties of bounded lattices which are not universal, to say nothing of finite-to-finite universal.

But what of semigroups, the algebras of primary interest to Sapir when he introduced the notion of  $Q$ -universal? Since every finite semigroup contains an idempotent element, no quasivariety of semigroups is finite-to-finite universal. That having been said, universal varieties of semigroups were characterized by Koubek and Sichler in [46]. Relatively universal varieties of semigroups were considered by Demlová and Koubek in a remarkable series of papers [17], [18], [19], and [20]. Of particular interest to them were idempotent semigroups, the  $Q$ -universality of which was subsequently considered by Adams and Dziobiak [7] and Sapir (private communication —see [7]). Demlová and Koubek went on to consider  $Q$ -universality for varieties of semigroups in [21] and [22], in the course of which they also address the question of whether, for a  $Q$ -universal quasivariety  $\mathbf{K}$  of semigroups,  $I(F)$  need be isomorphic to a sublattice of  $L(\mathbf{K})$ .

Could the hypothesis that a quasivariety be finite-to-finite universal be weakened, but still lead to it being  $Q$ -universal —see Problem 20 of [2]?

Koubek and Sichler [47] gave an example showing one way in which it could not. It was however conjectured that the hypothesis finite-to-finite universal could be weakened to finite-to-finite almost universal. In [49], Koubek and Sichler characterized the finitely generated varieties of 0-lattices which are finite-to-finite almost universal and, in [48], they showed that a variety of modular 0-lattices is finite-to-finite almost universal iff it is  $Q$ -universal. Subsequently [50], they have gone on to show that any finite-to-finite almost universal quasivariety is  $Q$ -universal, thereby verifying the conjecture.

## 6. AN INTRIGUING VARIETY: MONADIC BOOLEAN ALGEBRAS

To the newcomer, the technical nature of papers in this area can be somewhat daunting, if not overwhelming. Keeping this in mind, we conclude with an open problem that is readily accessible and requires little background —enter monadic Boolean algebras.

A *quantifier* on a Boolean algebra  $(B, \vee, \wedge, *, 0, 1)$  is a unary operation  $\nabla$  on  $B$  such that, for  $x$  and  $y \in B$ ,  $\nabla 0 = 0$ ,  $x \wedge \nabla x = x$ , and  $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$  (as shown by Halmos [40], it follows, for example, that  $\nabla \nabla x = \nabla x$  and  $\nabla(x \vee y) = \nabla x \vee \nabla y$ ). A *monadic Boolean algebra*  $(B, \vee, \wedge, *, \nabla, 0, 1)$  is a Boolean algebra  $(B, \vee, \wedge, *, 0, 1)$  with a quantifier  $\nabla$ . The variety of monadic Boolean algebras  $\mathbf{M}$  was introduced by Halmos in [40].

As shown by Monk [59], similar to the variety of pseudocomplemented distributive lattices, the lattice of subvarieties of  $\mathbf{M}$  is an  $\omega + 1$  chain

$$\mathbf{M}_{-1} \subset \mathbf{M}_0 \subset \mathbf{M}_1 \subset \dots \subset \mathbf{M}_n \subset \dots \subset \mathbf{M},$$

where  $\mathbf{M}_{-1}$  is the trivial variety and  $\mathbf{M}_0$  corresponds to the variety of Boolean algebras.

Since, for any  $B \in \mathbf{M}$ ,  $\text{End}(B) \not\cong C_3$  the cyclic group of order three, it follows that  $\mathbf{M}$  is not universal. Neither is  $\mathbf{M}$   $Q$ -universal — $L(\mathbf{M})$  is a countable lattice (see Adams and Dziobiak [4]). Why then are monadic Boolean algebras of any interest in this context?



Any universal quasivariety contains a proper class of non-isomorphic algebras each of which has only the identity as an endomorphism —so called *rigid* algebras. A long standing conjecture was whether any quasivariety containing a proper class of non-isomorphic rigid algebras is necessarily universal. Monadic Boolean algebras were the first known counterexample to this conjecture. However, even though  $\mathbf{M}$  contains a proper class of non-isomorphic monadic Boolean algebras, every rigid monadic Boolean algebra in  $\mathbf{M}_n$  is necessarily trivial. Moreover, for non-trivial *finite* monadic Boolean algebras  $B_0$  and  $B_1 \in \mathbf{M}$ , if  $\text{End}(B_0) \cong \text{End}(B_1)$ , then  $B_0 \cong B_1$ ; that is, finite monadic Boolean algebras are recoverable from their endomorphism monoids. Recalling that all algebras, not just finite algebras, in the variety of Boolean algebras  $\mathbf{M}_0$  are recoverable from their endomorphism monoids, prompts one to ask after the varieties  $\mathbf{M}_n$  for  $n < \omega$ :

Does there exist, for every  $n < \omega$ , some  $m_n < \omega$  such that monadic Boolean algebras in  $\mathbf{M}_n$  are recoverable up to one of  $m_n$  non-isomorphic algebras? Perhaps an even stronger statement is possible? Does there exist an  $m < \omega$  such that, for  $n < \omega$ , monadic Boolean algebras in  $\mathbf{M}_n$  are recoverable up to one of  $m$  non-isomorphic algebras?

For a fuller discussion of this rather enigmatic variety, see Adams and Dziobiak [8].

## REFERENCES

- [1] M.E.Adams, *Implicational classes of pseudocomplemented distributive lattices*, J. London Math. Soc. (2) 13 (1976), 381–384.
- [2] M.E.Adams, K.V.Adaricheva, W.Dziobiak, and A.V.Kravchenko, *Open questions related to the problem of Birkhoff and Maltsev*, Studia Logica 78 (2004), 357–378.
- [3] M.E.Adams and W.Dziobiak, *Q-universal quasivarieties of algebras*, Proc. Amer. Math. Soc. 120 (1994), 1053–1059.
- [4] M.E.Adams and W.Dziobiak, *Quasivarieties of distributive lattices with a quantifier*, Discrete Math. 135 (1994), 15–28.
- [5] M.E.Adams and W.Dziobiak, *Finite-to-finite universal quasivarieties are Q-universal*, Algebra Universalis 46 (2001), 253–283.
- [6] M.E.Adams and W.Dziobiak, *Q-universal varieties of bounded lattices*, Algebra Universalis 48 (2002), 333–356.
- [7] M.E.Adams and W.Dziobiak, *Quasivarieties of idempotent semigroups*, Internat. J. Algebra Comput. 13 (2003), 733–752.
- [8] M.E.Adams and W.Dziobiak, *Endomorphisms of monadic Boolean algebras*, Algebra Universalis 57 (2007), 131–142.
- [9] M.E.Adams, V.Koubek, and J.Sichler, *Pseudocomplemented distributive lattices with small endomorphism monoids*, Bull. Austral. Math. Soc. 28 (1983), 305–318.
- [10] M.E.Adams, V.Koubek, and J.Sichler, *Homomorphisms and endomorphisms in varieties of pseudocomplemented distributive lattices (with applications to Heyting algebras)*, Trans. Amer. Math. Soc. 285 (1984), 57–79.
- [11] M.E.Adams and J.Sichler, *Cover set lattices*, Canad. J. Math. 32 (1980), 1177–1205.
- [12] K.V.Adaricheva, W.Dziobiak, and V.A.Gorbunov, *Algebraic atomistic lattices of quasivarieties*, Algebra and Logic 36 (1997), 213–225.
- [13] K.V.Adaricheva and V.A.Gorbunov, *Equational closure operator and forbidden semidistributive lattices*, Siberian Math. J. 30 (1989), 831–849.
- [14] K.V.Adaricheva, V.A.Gorbunov, and V.I.Tumanov, *Join-semidistributive lattices and convex geometries*, Adv. Math. 173 (2003), 1–49.
- [15] K.V.Adaricheva and J.B.Nation, *Lattices of quasi-equational theories as congruence lattices of semilattices with operators*, to appear.

- [16] M.Armbrust and J.Schmidt, *Zum Cayleyschen Darstellungssatz*, (German) Math. Ann. 154 (1964), 70–72.
- [17] M.Demlová and V.Koubek, *Endomorphism monoids of bands*, Semigroup Forum 38 (1989), 305–329.
- [18] M.Demlová and V.Koubek, *Endomorphism monoids in small varieties of bands*, Acta Sci. Math. (Szeged) 55 (1991), 9–20.
- [19] M.Demlová and V.Koubek, *Endomorphism monoids in varieties of bands*, Acta Sci. Math. (Szeged) 66 (2000), 477–516.
- [20] M.Demlová and V.Koubek, *Weaker universalities in semigroup varieties*, Novi Sad J. Math. 34 (2004), 37–86.
- [21] M.Demlová and V.Koubek, *Weak alg-universality and  $Q$ -universality of semigroup quasivarieties*, Comment. Math. Univ. Carolin. 46 (2005), 257–279.
- [22] M.Demlová and V.Koubek, *On universality of semigroup varieties*, Arch. Math. (Brno) 42 (2006), 357–386.
- [23] W.Dziobiak, *On subquasivariety lattices of some varieties related with distributive  $p$ -algebras*, Algebra Universalis 21 (1985), 62–67.
- [24] W.Dziobiak, *On lattice identities satisfied in subquasivariety lattices of varieties of modular lattices*. Algebra Universalis 22 (1986), 205–214.
- [25] W.Dziobiak, *On atoms in the lattice of quasivarieties*, Algebra Universalis 24 (1987), 32–35.
- [26] G.Birkhoff, *On groups of automorphisms*, (Spanish) Revista Unión Mat. Argentina 11 (1946), 155–157.
- [27] R.Frucht, *Lattices with a given abstract group of automorphisms*, Canadian J. Math. 2 (1950), 417–419.
- [28] R.Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, (German) Compositio Math. 6 (1939), 239–250.
- [29] P.Goralčík, V.Koubek, and J.Sichler, *Universal varieties of  $(0, 1)$ -lattices*, Canad. J. Math. 42 (1990), 470–490.
- [30] V.A.Gorbunov, *Lattices of quasivarieties*, Algebra and Logic 15 (1976), 275–288.
- [31] V.A.Gorbunov, *The structure of lattices of varieties and lattices of quasivarieties: similarity and difference. II*, Algebra and Logic 34 (1995), 203–218
- [32] V.A.Gorbunov, **Algebraic theory of quasivarieties**, Plenum Publishing Co., New York, 1998.
- [33] V.A.Gorbunov and V.I.Tumanov, *A class of lattices of quasivarieties*, Algebra and Logic 19 (1980), 38–52.
- [34] G.Grätzer, **Universal algebra**, Van Nostrand Co., Princeton, Toronto, London, 1968.
- [35] G.Grätzer, **Lattice theory. First concepts and distributive lattices**, W. H. Freeman and Co., San Francisco, Calif., 1971.
- [36] G.Grätzer and H.Lakser, *The structure of pseudocomplemented distributive lattices. II. Congruence extension and amalgamation*, Trans. Amer. Math. Soc. 156 (1971), 343–358.
- [37] G.Grätzer and H.Lakser, *The structure of pseudocomplemented distributive lattices. III. Injective and absolute subretracts*, Trans. Amer. Math. Soc. 169 (1972), 475–487.
- [38] G.Grätzer, H.Lakser, and R.W.Quackenbush, *On the lattice of quasivarieties of distributive lattices with pseudocomplementation*, Acta Sci. Math. (Szeged) 42 (1980), 257–263.
- [39] G.Grätzer and J.Sichler, *On the endomorphism semigroup (and category) of bounded lattices*, Pacific J. Math. 35 (1970), 639–647.
- [40] P.R.Halmos, *Algebraic logic, I. Monadic Boolean algebras*, Compositio Math. 12 (1956), 217–249.
- [41] Z.Hedrlín and A.Pultr, *On full embeddings of categories of algebras*, Illinois J. Math. 10 (1966), 392–406.
- [42] Z.Hedrlín and J.Sichler, *Any boundable binding category contains a proper class of mutually disjoint copies of itself*, Algebra Universalis 1 (1971), 97–103.
- [43] P.Hell, *Full embeddings into some categories of graphs*, Algebra Universalis 2 (1972), 129–141.
- [44] P.Hell and J.Nešetřil, **Graphs and homomorphisms**, Oxford Lecture Series in Mathematics and its Applications 28, Oxford University Press, Oxford, 2004.
- [45] V.I.Igošin, *Review A384*, Ref. Zh. Mat. 2 (1976).
- [46] V.Koubek and J.Sichler, *Universal varieties of semigroups*, J. Austral. Math. Soc. Ser. A 36 (1984), 143–152.
- [47] V.Koubek and J.Sichler, *On relative universality and  $Q$ -universality*, Studia Logica 78 (2004), 279–291.
- [48] V.Koubek and J.Sichler, *Almost  $ff$ -universal and  $Q$ -universal varieties of modular 0-lattices*, Colloq. Math. 101 (2004), 161–182.
- [49] V.Koubek and J.Sichler, *Finitely generated almost universal varieties of 0-lattices*, Comment. Math. Univ. Carolin. 46 (2005), 301–325.
- [50] V.Koubek and J.Sichler, *Almost  $ff$ -universality implies  $Q$ -universality*, Appl. Categ. Structures, to appear.

- [51] A.V.Kravchenko, *On the lattice complexity of quasivarieties of graphs and endographs*, Algebra and Logic 36 (1997), 164–168.
- [52] A.V.Kravchenko, *Complexity of quasivariety lattices for varieties of unary algebras*, Siberian Adv. Math. 12 (2002), 63–76.
- [53] A.V.Kravchenko,  *$Q$ -universal quasivarieties of graphs*, Algebra and Logic 41 (2002), 173–181.
- [54] H.Lakser, *The structure of pseudocomplemented distributive lattices. I. Subdirect decomposition*, Trans. Amer. Math. Soc. 156 (1971), 335–342.
- [55] K.B.Lee, *Equational classes of distributive pseudo-complemented lattices*, Canad. J. Math. 22 (1970), 881–891.
- [56] K.D.Magill, *The semigroup of endomorphisms of a Boolean ring*, J. Austral. Math. Soc. 11 (1970), 411–416.
- [57] R.McKenzie, *On minimal simple lattices*, Algebra Universalis 32 (1994), 63–103.
- [58] E.Mendelsohn, *On a technique for representing semigroups as endomorphism semigroups of graphs with given properties*, Semigroup Forum 4 (1972), 283–294.
- [59] J.D.Monk, *On equational classes of algebraic versions of logic. I*, Math. Scand. 27 (1970), 53–71.
- [60] A.Pultr, *Concerning universal categories*, Comment. Math. Univ. Carolinae 5 (1964), 227–239.
- [61] A.Pultr and J.Sichler, *Primitive classes of algebras with two unary idempotent operations, containing all algebraic categories as full subcategories*, Comment. Math. Univ. Carolinae 10 (1969), 425–445.
- [62] A.Pultr and V.Trnková, **Combinatorial, algebraic and topological representations of groups, semigroups and categories**, North-Holland, Amsterdam, 1980.
- [63] P.Ribenboim, *Characterization of the sup-complement in a distributive lattice with last element*, Summa Brasil. Math. 2, (1949), 43–49.
- [64] M.V.Sapir, **Quasivarieties of semigroups**, Ph.D. Thesis, Svedlovsk, 1980.
- [65] M.V.Sapir, *The lattice of quasivarieties of semigroups*, Algebra Universalis 21 (1985), 172–180.
- [66] B.M.Schein, *Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. 68 (1970), 31–50.
- [67] M.V.Semenova, *Lattices of suborders*, Siberian Math. J. 40 (1999), 577–584.
- [68] J.Sichler, *Concerning minimal primitive classes of algebras containing any category of algebras as a full subcategory*, Comment. Math. Univ. Carolinae 9 (1968), 627–635.
- [69] J.Sichler, *Nonconstant endomorphisms of lattices*, Proc. Amer. Math. Soc. 34 (1972), 67–70.
- [70] J.Sichler, *Testing categories and strong universality*, Canad. J. Math. 25 (1973), 370–385.
- [71] J.Sichler, *Group-universal unary varieties*, Algebra Universalis 11 (1980), 12–21.
- [72] S.V.Sizyĭ, *Quasivarieties of graphs*, Siberian Math. J. 35 (1994), 783–794.
- [73] M.P.Tropin, *An embedding of a free lattice into the lattice of quasivarieties of distributive lattices with pseudocomplementation*, Algebra and Logic 22 (1983), 113–119.
- [74] V.I.Tumanov, *Finite distributive lattices of quasivarieties*, Algebra and Logic 22 (1983), 168–181.
- [75] P.Vopěnka, A.Pultr, and Z.Hedrlín, *A rigid relation exists on any set*, Comment. Math. Univ. Carolinae 6 (1965), 149–155.
- [76] A.Wroński, *The number of quasivarieties of distributive lattices with pseudocomplementation*, Polish. Acad. Sci. Inst. Philos. Social. Bull. Sect. Logic 5 (1976), 115–121.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT NEW PALTZ, NEW PALTZ, NY 12561

*E-mail:* adamsm@newpaltz.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUERTO RICO, MAYAGÜEZ, PR 00681

*E-mail:* w.dziobiak@gmail.com