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GEOMETRY OF THE EUCLIDEAN HAMILTONIAN SUBOPTIMAL AND OPTIMAL PATHS IN THE $\mathscr{N}(K_n(\sqrt[n]{1}),(d_{ij})_{n\times n})$'S NETWORKS

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ABSTRACT. The results concern the study of hamiltonian cycles and paths in the network built by the complete graph K_n with vertices on the n-th roots of the unity and with the euclidean distances between nodes. The first aim is to single out and enumerate the reflective euclidean hamiltonian cycles whereas the second one is to find the optimum cycles and paths. A postulate of the geometric optics allows us to identify the shortest euclidean hamiltonian cycles while the longest cycles only at odd cases. In even instances, the networks with $4p$ vertices have the stargons of maximum density, however the longest cycles do not belong to the reflective cycles. We solve, by a geometric proposal, the $\frac{n}{2}$ longest euclidean hamiltonian path problems and we find a representative of the longest cycles.

1. INTRODUCTION

Let us consider the network $\mathscr{N}(K_n(\sqrt[n]{1}),D)$, where $K_n(\sqrt[n]{1})$ is the complete graph with vertices on the n-th roots of the unity, and $D=(d_{ij})$ is the $n\times n$ matrix of the euclidean distances between nodes [12, 13]. We deal with the *optimum euclidean hamiltonian cyclic and non-cyclic paths* [6]. The search for the euclidean hamiltonian optimum involves tougher computational tasks since the network with its K_n graph architecture has $\frac{(n-1)!}{2}$ hamiltonian cycles [3, 8]. Even for a moderate number of nodes n, checking all such cycles would be ludicrous as we had experienced in simulations performed in 1998 [12] and 2005[1]. Thus other approaches are called for, known as approximation techniques [1, 5]. Barnivok in [1] yields a polynomial time approximation algorithm for a special version of the longest hamiltonian circuit in a euclidean space, which is applicable to our context. Fekete also provides an algorithm in polynomial time for computing the length of the longest tour of a set of points in the plane with rectilinear distances. Moreover, this author proved that *the Max TSP under euclidean norm in* $\mathbb{R}^d$ *is* $\mathscr{NP}$*-hard for any fixed* $d\geq 3$ and herein [5] in p. 345, leaves an open conjecture about the complexity status of the longest cycles for euclidean distances in the plane. We apply a simple approximation method which uses local greedy and anti-greedy algorithmic strategies —which are known as the nearest and farthest neighbor method [2]. Greedy and anti-greedy techniques attain the shortest cycles in $\mathscr{N}(K_n(\sqrt[n]{1}),D)$'s networks and the longest cycles in $\mathscr{N}(K_{n=2p+1}(\sqrt[2p+1]{1}),(d_{ij})_{n\times n})$'s

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[1]Simulations performed with a Pentium 4, CPU speed 2.4 GHz, 256 MB of RAM, Windows XP and DevC$_{++}$ ver. 4.9.9.2 software. The processing time was in the networks with 14 nodes less than one second in a brute force checking, meanwhile with some clever programming behind the exploration ran two seconds over 16 nodes and 6 days and four hours on 26 nodes. Last data forecast for 42 nodes the laughable requirement of 13 millions of years. Acknowledgment to Agustín Claverie, Mathematics laboratory, U.N.S.

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networks. These optimal hamiltonian pathways are baited by the lure of the perfect or regular form, therefore, we select an adequate theoretical context from the geometric optics scope at the spherical mirror circumstances [7] in order to search for a mathematical comprehension. On the contrary, the cycle rendered up by the anti-greedy exploration, in $\mathcal{N}(K_{n=2p+2}(\sqrt[2p+2]{1}),(d_{ij})_{n\times n})$'s networks, is far from the longest cycle (e.g. refer to first and last pictures in Figure 3, p. 83). The non regular shape of the anti-greedy sub-optimal cycle inspired us with the construction of the configurations that resolve the $\frac{n}{2}$ different longest euclidean non-cyclic path problems and finally confirmed a representative of the longest cycles in these networks. Concretely, we confirm the shortest and longest euclidean hamiltonian cycles in $\mathcal{N}(K_n(\sqrt[n]{1}),(d_{ij})_{\mathrm{nxn}})$'s networks. We solve the $\frac{n}{2}$ different longest euclidean hamiltonian path problems in the $\mathcal{N}(K_{n=2p+2}(\sqrt[2p+2]{1}),(d_{ij})_{n\times n})$'s networks as by-products.

This work contains two essential sections, Section 2 and 3, the Conclusion and the Appendices A, B and C. Appendices A and B have the demonstrations of *Theorem 2.1.1* and *Theorem 3.1.2*, postponed in order to put together the aim of this contribution. In Section 2, the shortest cycles in the $\mathcal{N}(K_n(\sqrt[n]{1}),(d_{ij})_{n\times n})$'s networks and the longest cycles in the $\mathcal{N}(K_{n=2p+1}(\sqrt[2p+1]{1}),(d_{ij})_{n\times n})$'s networks are confirmed, resting on the assumption of the geometric optics theory. Therein, the reflective cycles in $\mathcal{N}(K_n(\sqrt[n]{1}),(d_{ij})_{n\times n})$'s networks are enumerated. Meanwhile, Section 3 is devoted to find the longest euclidean hamiltonian cycles in the $\mathcal{N}(K_{n=2p+2}(\sqrt[2p+2]{1}),(d_{ij})_{n\times n})$'s networks. Herein, in a chain lengths from the anti-greedy suboptimal to the longest euclidean hamiltonian cycles, if they exist —Appendix C— the reflective cycles that look like the stargons of maximum density, are located.

2. First main result

We select a theoretical context from Hamilton's ideas on geometric optics circumstances at the spherical mirror. Precisely, underneath the nowadays postulate of *"A light ray, in going between two points, must traverse the optical path which has stationary length with respect to variations of the path"* [19, 10, 14], we built a mathematical model that confirms at the spherical mirror [7], (equiv. for us, to the unitary circle) that reflective pathways may have maxima, minima and no optima their travelled lengths [18]. This particular phenomenon sires certain knowledge about the reflective suboptimal and optimal euclidean hamiltonian paths in $\mathcal{N}(K_n(\sqrt[n]{1}),(d_{ij})_{n\times n})$'s networks [15].

We choose the geometric paths that start up at $C=(-1,0)$ of the spherical mirror of unitary radius, touch n times —including the last touching— anywhere on the hollowed mirror, and end up at $B=(\cos\beta,\sin\beta)$, with $-\pi\le\beta\le 0$.

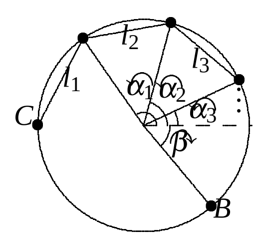

FIGURE 1. Measure of α_i's parameter.

2.1. **Stationary length paths on the unitary circle (unconstrained geometric paths).**

In this geometry each n array of angles $(\alpha_1,\ldots,\alpha_{n-1},\beta)$, denoted (α_i,β), determines a path with $n+1$ vertices —including the initial and arrival points— and n linear branches.

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This path may have two or more coincident vertices and linear branches shrunk to a point. For each $\beta \in [-\pi, 0]$ the $n-1$ angles $\alpha_i \in \mathbb{R}$ are selected (see Figure 1) as independent variables of the overall travelled length function of the paths $F_n(\alpha_i, \beta)$.

The length of the geometric path determined by (α_i, β) is then

$$F_n(\alpha_i,\beta) = \sqrt{2}\left\{\sqrt{1+\cos\alpha_1} + \sum_{i=2}^{n-1}\sqrt{1-\cos(\alpha_i-\alpha_{i-1})} + \sqrt{1-\cos(\beta-\alpha_{n-1})}\right\}. \quad (1)$$

$F_n(\alpha_i, \beta)$ is a continuous function everywhere. Furthermore $F_n(\alpha_i, \beta) = F_n(\alpha_i + 2k_i\pi, \beta)$ for any $k_i \in \mathbb{Z}$, hence over the compact set $\mathscr{K} = [0, 2\pi]^{n-1} \times [-\pi, 0]$, $F_n(\alpha_i, \beta)$ attains all the values of its image.

Observation 2.1.1. *The case $\beta = 0$ deals with non-cyclic paths, but its peculiar characteristics are outside the interests of our present research* [17]. *When $\beta = -\pi$, $(B \equiv C)$, for any polygonal cyclic trajectory, there is an n-array $(\alpha_1, \cdots, \alpha_{n-1}, -\pi)$ which characterizes them, meanwhile $F_n(\alpha_1, \cdots, \alpha_{n-1}, -\pi)$ is the length of those cyclic paths. In particular, amongst these pathways are those that have as vertices the $e^{\pi i}\sqrt[m]{1}$'s points, with $m \leq n$.*

We recognize as a reflective —cyclic or non cyclic— trajectory, that which in each vertex verifies the "reflection law", under the approximation region of the geometric optics.

Theorem 2.1.1. *For each β, $-\pi < \beta \leq 0$, $F_n(\alpha_i, \beta)$ has n stationary critic points $\alpha_{c_i}(k)$:*

$$\alpha_{c_i}(k) = \frac{(n-i)\pi + i\beta}{n} + i.\frac{2k\pi}{n}, \quad k = 0,1,2,\ldots,n-1; \quad i = 1,2,\ldots,n-1. \quad (2)$$

i) *If $-\pi < \beta \leq 0$, each stationary critic point evolves one reflective trajectory of n-linear branches whose length is $F_n(\alpha_{c_i}(k), \beta) = \sqrt{2}n\sqrt{1-\cos\Delta\alpha_{c_i}(k)}$, where $\Delta\alpha_{c_i}(k) = \alpha_{c_{i-1}}(k) - \alpha_{c_i}(k)$.*

ii) *If $\beta = -\pi$ is $C \equiv B$ and $F_n(\alpha_i, -\pi)$ has $n-1$ stationary critic points $\alpha_{c_i}(k)$ given by (2) but with $k = 0, 2, \cdots, n-1$.*

iii) *All the stationary critic points of $F_n(\alpha_i, \beta)$ are relative maxima.*

iv) *The singular critic points, wherever $F_n(\alpha_i, \beta)$ is a non-differentiable function, evolve trajectories with less than $n-1$ linear branches.*

v) *The minimum of $F_n(\alpha_i, \beta)$ is the distance between C and B.*

vi) *The maximum of $F_n(\alpha_i, \beta)$ is the longest of the relative maxima.*
If n is odd, $\max(F_n(\alpha_i,\beta)) = n\sqrt{2}\sqrt{1+\cos(\frac{\beta}{n})} = F_n(\alpha_{c_i}(\frac{n+1}{2}), \beta)$.
If n is even, $\max(F_n(\alpha_i,\beta)) = n\sqrt{2}\sqrt{1+\cos(\frac{\beta+\pi}{n})} = F_n(\alpha_{c_i}(\frac{n}{2}+1), \beta)$.

vii) *When $-\pi < \beta < 0$ and n is even, the lengths of the n reflective trajectories (all different) are ordered in a strictly increasing chain of inequalities:*

$$0 < F_n(\alpha_{c_i}(1);\beta) < F_n(\alpha_{c_i}(0);\beta) < F_n(\alpha_{c_i}(2);\beta) < F_n(\alpha_{c_i}(n-1);\beta) < F_n(\alpha_{c_i}(3);\beta) <$$
$$< \underbrace{F_n(\alpha_{c_i}(n-j);\beta) < F_n(\alpha_{c_i}(j+2);\beta)}_{2\leq j\leq \frac{n}{2}-2} < F_n(\alpha_{c_i}(\frac{n}{2}+1);\beta) < 2n. \quad (3)$$

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When $\beta = -\pi$ and n is even:

$$0 = F_n(\alpha_{c_i}(1);-\pi) < F_n(\alpha_{c_i}(0);-\pi) = F_n(\alpha_{c_i}(2);-\pi) < F_n(\alpha_{c_i}(n-1);-\pi) = F_n(\alpha_{c_i}(3);-\pi) < \\ < \underbrace{F_n(\alpha_{c_i}(n-j);-\pi) = F_n(\alpha_{c_i}(j+2);-\pi)}_{2\le j\le \frac{n}{2}-2} < F_n(\alpha_{c_i}(\frac{n}{2}+1);-\pi) = 2n. \tag{4}$$

When $\beta = 0$ and n is even:

$$0 < F_n(\alpha_{c_i}(1);0) = F_n(\alpha_{c_i}(0);0) < F_n(\alpha_{c_i}(2);0) = F_n(\alpha_{c_i}(n-1);0) < \\ < \underbrace{F_n(\alpha_{c_i}(j);0) = F_n(\alpha_{c_i}(n+1-j);0)}_{3\le j\le \frac{n}{2}} < 2n. \tag{5}$$

viii) *On the other hand, if n is odd and $-\pi < \beta < 0$ the increasing lengths of the n reflective trajectories are:*

$$0 < F_n(\alpha_{c_i}(1);\beta) < F_n(\alpha_{c_i}(0);\beta) < F_n(\alpha_{c_i}(2);\beta) < \underbrace{F_n(\alpha_{c_i}(n-j);\beta) < F_n(\alpha_{c_i}(j+2);\beta)}_{1\le j\le \frac{n-3}{2}} < 2n. \tag{6}$$

If $\beta = -\pi$ and n is odd:

$$0 = F_n(\alpha_{c_i}(1);-\pi) < F_n(\alpha_{c_i}(0);-\pi) = F_n(\alpha_{c_i}(2);-\pi) < \underbrace{F_n(\alpha_{c_i}(n-j);-\pi) = F_n(\alpha_{c_i}(j+2);-\pi)}_{1\le j\le \frac{n-3}{2}} < 2n. \tag{7}$$

If $\beta = 0$ and n is odd:

$$0 < F_n(\alpha_{c_i}(1);0) = F_n(\alpha_{c_i}(0);0) < \underbrace{F_n(\alpha_{c_i}(j);0) = F_n(\alpha_{c_i}(n+1-j);0)}_{2\le j\le \frac{n-1}{2}} < F_n(\alpha_{c_i}(\frac{n+1}{2});0) = 2n. \tag{8}$$

ix) *The reflective pathways travel in ccw. or cw. circulation, according to the following scheme,*

- *when $-\pi < \beta < 0$*

$$\begin{array}{llll} n \text{ even:} & \curvearrowleft \text{ if } 1 \le k < \frac{n+1}{2}; \text{ and} & \curvearrowright \text{ if } \frac{n}{2}+1 \le k \le n & (k=0\equiv k=n) \\ n \text{ odd:} & \curvearrowleft \text{ if } 1 \le k \le \frac{n+1}{2}; \text{ and} & \curvearrowright \text{ if } \frac{n+1}{2}+1 \le k \le n & (k=0\equiv k=n) \end{array} \tag{9}$$

- *when $\beta = -\pi$*

$$\begin{array}{llll} n \text{ even:} & \curvearrowleft \text{ if } 1 < k \le \frac{n}{2}+1; \text{ and} & \curvearrowright \text{ if } \frac{n}{2}+2 \le k \le n & (k=0\equiv k=n) \\ n \text{ odd:} & \curvearrowleft \text{ if } 1 < k \le \frac{n+1}{2}; \text{ and} & \curvearrowright \text{ if } \frac{n+1}{2}+1 \le k \le n & (k=0\equiv k=n) \end{array} \tag{10}$$

- *when $\beta = 0$*

$$\begin{array}{llll} n \text{ even:} & \curvearrowleft \text{ if } 1 \le k \le \frac{n}{2}; \text{ and} & \curvearrowright \text{ if } \frac{n}{2}+1 \le k \le n & (k=0\equiv k=n) \\ n \text{ odd:} & \curvearrowleft \text{ if } 1 \le k \le \frac{n+1}{2}; \text{ and} & \curvearrowright \text{ if } \frac{n+1}{2}+1 \le k \le n & (k=0\equiv k=n) \end{array} \tag{11}$$

Proof. Refer to Appendix A, from page 78 to 80. ■

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2.1.1. *The reflective pathways become cycles on the* $e^{\pi i}\sqrt[n]{1}$*'s points when* $\beta = -\pi$. In this particular case the length of the geometric paths is given by

$$F_n(\alpha_i, -\pi) = \sqrt{2}\sum_{i=1}^{n}\sqrt{1-\cos(\alpha_{i-1}-\alpha_i)} \qquad (\alpha_0 = \pi \text{ and } \alpha_n = \beta = -\pi).$$

The stationary critic points of $F_n(\alpha_i, -\pi)$ build $n-1$ reflective pathways with n linear branches and vertices on the set of points $e^{\pi i}\sqrt[n]{1} : \{V_0, \cdots, V_{n-1}\}$.

After agreements of terms in the $e^{\pi i}\sqrt[n]{1}$'s set of points we deal with the peculiar characteristics of this case.

Let n be any natural number and let us consider the n-regular polygon with vertices in the set of n points of the form $e^{\pi i}\sqrt[n]{1}$, clockwise numbered, $V_0, \cdots, V_{n-1}$, from $V_0 = (-1, 0)$.

Let $l_{\max}$ be the diameter, it joins the vertex V_j with its opposite $V_{j+\frac{n}{2}}$, only if n is even. If $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$, let L_k^- represent the segment that links the vertices V_j with V_{j+k} and L_k^+ the segment that connects V_j with V_{n+j-k}, although $l^-_{\text{q.max}}$ and $l^+_{\text{q.max}}$ designate the quasi-diameter segments $L^-_{\lfloor \frac{n}{2} \rfloor -1}$ and $L^+_{\lfloor \frac{n}{2} \rfloor -1}$ respectively. The subindices are added module n. The arrival vertex V_{j+k} when L_k^- is traced from the vertex V_j begets a clockwise angular advance of $k\dfrac{2\pi}{n}$. On the other hand, if L_k^+ is traced from the same vertex the corresponding clockwise angular variation is $\left(1-\dfrac{k}{n}\right)\dfrac{2\pi}{n}$. For example, in Figure 2 (right), the case $j = 0$ for n even is illustrated.

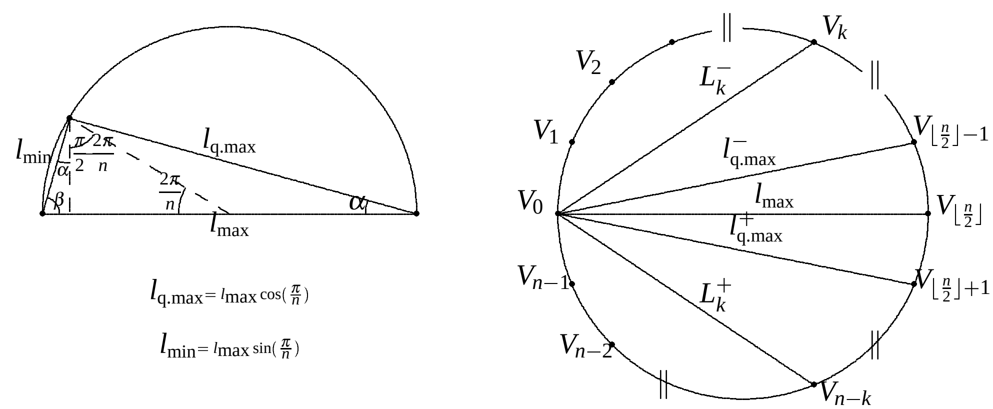

FIGURE 2. Expressions of $l_{q.\max}$ and $l_{\min}$ (left). Diagram of the L_k's from V_0

Notation: Let T_k^- denote a reflective cycle built by n sides L_k^- and with vertices in $\{V_0, \cdots, V_{n-1}\}$, while T_k^+ is formed by n sides L_k^+, and T_k is the length $\mathfrak{L}(T_k^-) = \mathfrak{L}(T_k^+)$. In this circumstance, $T_k = 2n\cos\left[\left(\frac{n}{2}-k\right)\frac{\pi}{n}\right]$.

Lemma 2.1.1. *The stationary critic points of* $F_n(\alpha_i, -\pi)$ *correspond to reflective cycles built by n sides* L_k*. Some of them are hamiltonian cycles of order n over the set* $e^{\pi i}\sqrt[n]{1}$*. Their lengths are ordered in the chain of inequalities (4) if n is even and (7) if n is odd, in Theorem 2.1.1.*

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Proof. Each one of the trajectories T_k^- and T_k^+ corresponds to a stationary critic point, in accordance with the following details:

n even:

$$T_k^- \longleftrightarrow \alpha_{c_i}(n+1-k),\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$$

$$T_k^+ \longleftrightarrow \alpha_{c_i}(k+1),\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$$

The angular coordinates $\alpha_{c_i}(\frac{n}{2}+1)$ are associated to the $\frac{n}{2}$-digons —built by n sides $l_{\max}$— denoted as $T^\star_{\lfloor \frac{n}{2} \rfloor}$. It has been used that $\alpha_{c_i}(0)$ and $\alpha_{c_i}(n)$ characterize the same path T_1^-. Finally, $T_0^\star$ denotes the path which rebounds at $V_0 \equiv C$, which corresponds to the singular critic points $\alpha_{c_i}(1) = (\pi, \cdots, \pi)$.

n odd:

$$T_k^- \longleftrightarrow \alpha_{c_i}(n+1-k),\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$$

$$T_k^+ \longleftrightarrow \alpha_{c_i}(k+1),\ 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$$

$$T_0^\star \longleftrightarrow \alpha_{c_i}(1).$$

Therefore, the inequalities (4) and (7) (Theorem 2.1.1, p. 69) are respectively equivalent to:

$$0 = T_0^\star < T_1^- = T_1^+ < T_2^- = T_2^+ < \underbrace{T_{j+1}^- = T_{j+1}^+}_{2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 2} < T^\star_{\lfloor \frac{n}{2} \rfloor} = 2n \tag{12}$$

$$0 = T_0^\star < T_1^- = T_1^+ < \underbrace{T_{j+1}^- = T_{j+1}^+}_{1 \leq j \leq \frac{n-3}{2}} < 2n. \tag{13}$$

Obviously, both chains of inequalities are true, once they are rewritten in the form (12) and (13). ■

2.1.2. *Hamiltonian cyclic reflective paths in the $\mathcal{N}(K_n(e^{\pi i}\sqrt[n]{1}), (d_{ij})_{n\times n})$'s networks (constrained geometric paths).*

Theorem 2.1.2. *The unique —except orienteering— reflective hamiltonian cycles of order n in $\mathcal{N}(K_n(\sqrt[n]{1}), (d_{ij})_{n\times n})$'s networks, are n-stargons constructed with n sides L_k^-, for the following values of k:*

a_1) *$1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and k is relatively prime to n, if n is odd (the n-stargon of maximum density does exist, i.e. built by n sides $l_{q.max} = L_{\lfloor \frac{n}{2} \rfloor}$).*

a_2) *$1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ and k is relatively prime to n, if $n = 4p$ (the n-stargon of maximum density does exist, i.e. built by n sides $l_{q.max}$);*

$1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ and k is relatively prime to n, if $n = 2(2p+1)$ (the n-stargon of maximum density does not exist, i.e. n- $l_{q.max}$ does not traverse throughout the $e^{\pi i}\sqrt[n]{1}$'s points).

Proof.

a_1)

If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and the line L_k^- is drawn from the initial vertex V_0 to the arrival vertex V_k, the reflection at V_k requires the incorporation of another L_k^-. Since the drawing of the side L_k^- from any vertex generates a cw. angular advance of $k\frac{2\pi}{n}$, it is clear that the sequential addition of n sides L_k^- from V_0 ends up at V_0.

Since k and n have no common divisor, then m does not exist, $0 < m < n$, such that $m\frac{2k\pi}{n} = 2r\pi, r \in \mathbb{Z}$. Therefore, the vertex V_0 is reached only after the location of the n sides L_k^-.

However, if $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ but k and n have common divisor d, the cycle, constructed as was indicated above, arrives at V_0 after touching only $\frac{n}{d}$ vertices. In this case, the location of n sides L_k^- implies that the first cycle formed over the $\frac{n}{d}$ vertices is d times repeated, then this cycle does not belong to the hamiltonian cycles of order n. Consequently, a reflective hamiltonian cycle of order n over $\{V_0, \cdots, V_{n-1}\}$ must be an n-stargon built by n sides L_k^- (or L_k^+) for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and k relatively prime to n.

This result is independent from the evenness of n.

The stargon of n sides $L^-_{\lfloor \frac{n}{2} \rfloor}$ is a hamiltonian cycle of order n, since $m < n$ does not exist such that $m\lfloor \frac{n}{2} \rfloor \frac{2\pi}{n} = \left(m - \frac{m}{n}\right)\pi = 2r\pi,\ r \in \mathbb{Z}$.

a_2)

If n is even, $\lfloor \frac{n}{2} \rfloor$ divides n, therefore the reflective cycle built by n sides $L_{\lfloor \frac{n}{2} \rfloor} = l_{\text{q.max}}$ does not belong to the hamiltonian cycles.

If $n = 4p$, $m < n$ does not exist such that $m\left(\frac{n}{2} - 1\right)\frac{2\pi}{n}$ is a multiple of 2π, therefore it is a hamiltonian cycle of order n.

If $n = 2(2p+1)$, for $m = 2p+1$ results $m(\frac{n}{2} - 1)\frac{2\pi}{n} = 2p\pi$, therefore the cycle of n sides $L_{\lfloor \frac{n}{2} \rfloor - 1} = l_{\text{q.max}}$ does not belong to the hamiltonian cycles of order n. ■

Observation 2.1.2. *Euler's function* $\phi(n)$*, is the cardinality of the set* $\{m/gcd(m,n) = 1, 0 < m < n\}$*, enumerates the reflective hamiltonian cycles in* $\mathcal{N}(K_n(e^{\pi i}\sqrt[n]{1}), (d_{ij})_{n\times n})$*'s networks* [9]. $\phi(n)$ *reckoning both ccw. and cw. shapes.*

Example 2.1.1. *There are 22 reflective paths on* $\mathcal{N}(K_{n=23}(\sqrt[23]{1}), (d_{ij})_{23\times 23})$*'s networks, since* $\phi(23) = 22$*. From* $\phi(24) = 8$*, the reflective paths in the* $\mathcal{N}(K_{n=24}(\sqrt[24]{1}), (d_{ij})_{24\times 24})$*'s networks look like the stargons* $\{\{\frac{24}{1}\}, \{\frac{24}{5}\}, \{\frac{24}{7}\}, \{\frac{24}{11}\}\}$ *cw. and ccw. travelled*[2].

Corollary 2.1.1. *The shortest euclidean hamiltonian cycle in* $\mathcal{N}(K_n(\sqrt[n]{1}), (d_{ij})_{n\times n})$*'s networks and the longest euclidean hamiltonian cycle in* $\mathcal{N}(K_{2p+1}(\sqrt[2p+1]{1}), (d_{ij})_{n\times n})$*'s networks correspond to reflective paths.*

Proof. Obviously the shortest euclidean hamiltonian cycle in the $\mathcal{N}(K_n(\sqrt[n]{1}), (d_{ij})_{n\times n})$'s networks is the regular n-gon, whichever be the evenness of n.

On the other hand, if n is odd, we have shown that the longest euclidean hamiltonian cycle, with vertices in $\{V_0, \cdots, V_{n-1}\}$ is the reflective trajectory constructed by n sides $l_{\text{q.max}}$. Since the vertices $V_0, \cdots, V_{n-1}$ are simply a rotation of the n points $\sqrt[n]{1}$, the longest

[2] p-gon $\{p/d\}$ symbolizes some regular p-stargon of $\{p/d\}$ density. We adopt the notation first used by the Swiss mathematician L. Schaläfli (1814–1895) [4].

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euclidean hamiltonian cycles in $\mathcal{N}(K_{2p+1}(\sqrt[2p+1]{1}),(d_{ij})_{n\times n})$'s networks are the reflective paths built by n sides $l_{\text{q.max}}$, with nodes in the $\sqrt[n]{1}$'s points, i.e. the stargon of maximum density. ■

Observation 2.1.3. *Theorem 2.1.1 shows that the maximum of* $F_n(\alpha_{c_i}(\frac{n}{2}+1),-\pi)=2n$ *does not correspond to a hamiltonian cycle of order n, if n is even. Moreover Theorem 2.1.2 clarifies that if* $n\neq 4p$ *the stargon of maximum density does not exist. Consequently, both statements leave the door open to the searching for the longest cycles in* $\mathcal{N}(K_{2p+2}(\sqrt[2p+2]{1}),(d_{ij})_{n\times n})$*'s networks amongst the non reflective paths. In the next section we struggle in this commitment, i.e. to solve the longest euclidean hamiltonian cycle problems in* $\mathcal{N}(K_{2p+2}(\sqrt[2p+2]{1}),(d_{ij})_{n\times n})$*'s networks.*

3. SECOND MAIN RESULT

The section is devoted to confirm the configurations that accomplish the longest euclidean hamiltonian cycles in $\mathcal{N}(K_{n=2p+2}(\sqrt[2p+2]{1}),D)$'s networks with $n\geq 6$. This achievement is carried out from Theorem 3.1.1 to 3.1.5.

The proof of *Theorem 3.1.2* is postponed to *Appendix B*. In this way, the backbone of the reasonings is not disrupted. *Appendix C* deals with the configurations that look like the star-polygons of max. density in $\mathcal{N}(K_{n=4p}(\sqrt[4p]{1}),D)$'s networks.

3.1. The backbone of the logical and geometrical thoughts.

Theorem 3.1.1. *Let* $\mathcal{N}=\{K_{n=2p+2}(\sqrt[2p+2]{1}),D\}$ *be the networks with their* $n=2p+2$ *nodes,* $V_0,\cdots,V_{n-1}$*, corresponding to the n-th roots of the unity, cw. ordered from* $V_0=(-1,0)$*. Therein exists a closed trajectory with n sides and vertices in the set* $\{V_0,\dots,V_{n-1}\}$*, it starts and ends up at* V_0*. For each k between* 1 *and* $\frac{n}{2}-1$*, this path is built by a single side* L_k^-*,* $\dfrac{n}{2}+1-k$ *diameters* l_{max} *and* $\dfrac{n}{2}+k-2$ *quasi-diameters* $l^{\pm}_{q.max}$*.*

Proof. From the vertex V_0, any reordering in sequence of the n segments L_k^-, L_k^+, $l_{\max}$, $l^-_{\text{q.max}}$ or $l^+_{\text{q.max}}$ determines a trajectory with its vertices in the set $\{V_0,\cdots,V_{n-1}\}$. Although, any of the mentioned reorderings could not build a closed path or neither could it pass through each vertex in the set $\{V_0,\cdots,V_{n-1}\}$. In general, both properties are false. The required trajectory should have exactly: one side L_k^-, $\dfrac{n}{2}+1-k$ sides $l_{\max}$, t sides $l^-_{\text{q.max}}$ and $\dfrac{n}{2}+k-2-t$ sides $l^+_{\text{q.max}}$. On account of the cw. angular advance as the result of passing from a vertex V_i to V_j after being added one of the sides previously mentioned, the existence of such closed paths required, for some $m\in\mathbb{N}$, the existence of an integer solution for the following equation:

$$\frac{n}{2}-(k-1)+(\frac{n}{2}+k-2-t)(1+\frac{2}{n})+t(1-\frac{2}{n})+k\frac{2}{n}=2m,\quad 0\leq t\leq\frac{n}{2}+k-2. \quad (14)$$

The single integer solution for (14) is $t=k-1$. ■

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Consequently, the existence of pathways built with initial and final vertex at V_0, a single side L_k^-, $(\frac{n}{2}+1-k)$ sides $l_{\max}$, $(k-1)$ sides $l^-_{\text{q.max}}$ and $(\frac{n}{2}-1)$ sides $l^+_{\text{q.max}}$ is already established. Furthermore, *Theorem 3.1.2* warrants the existence of the euclidean hamiltonian cyclic paths of order n built in that manner in $\mathscr{N}=\{K_{n=2p+2}(\sqrt[2p+2]{1}),D\}$'s networks.

Theorem 3.1.2. *Let $\mathscr{N}=\{K_{n=2p+2}(\sqrt[2p+2]{1}),D\}$ be the networks with their $n=2p+2$ nodes, $V_0,\cdots,V_{n-1}$, corresponding to the n-th roots of the unity clockwise ordered from $V_0=(-1,0)$.*

Each trajectory Γ_k, $1\le k\le \frac{n}{2}-1$, determined by the sequence of segments

$$\Gamma_k:\quad \underbrace{L_k^-,\ \underbrace{l^-_{q.max},\cdots,l^-_{q.max}}_{k-1},\ l_{max},\ \underbrace{l^+_{q.max},\cdots,l^+_{q.max}}_{k},}_{2k+1}\ \underbrace{l_{max},\ l^+_{q.max},\ l_{max},\cdots,l^+_{q.max},\ l_{max}}_{\left(\frac{n}{2}-k\right)l_{max}\ and\ \left(\frac{n}{2}-k-1\right)l^+_{q.max}} \tag{15}$$

is a euclidean hamiltonian cycle, of order n, which passes through the n vertices of the networks.

Proof. Refer to Appendix B, from page 80 to 82. ■

Theorem 3.1.3. *The overall lengths, $\mathfrak{L}(\Gamma_k)$, of the cyclic paths Γ_k, $1\le k\le\frac{n}{2}-1$, defined in Theorem 3.1.2, line up in a chain of strictly increasing inequalities with respect to k, i.e.,*

$$\mathfrak{L}(\Gamma_1)<\mathfrak{L}(\Gamma_2)<\cdots<\mathfrak{L}(\Gamma_{\frac{n}{2}-1}). \tag{16}$$

Proof. It is necessary to demonstrate that

$$\left(\frac{n}{2}-k+1\right)l_{\max}+\left(\frac{n}{2}+k-2\right)l_{\text{q.max}}+l_{\max}\sin\left(k\frac{\pi}{n}\right)$$
$$<\left(\frac{n}{2}-k\right)l_{\max}+\left(\frac{n}{2}+k-1\right)l_{\text{q.max}}+l_{\max}\sin\left[(k+1)\frac{\pi}{n}\right].$$

This inequality is true for $1\le k\le\dfrac{n}{2}-2$. Equivalently written as the inequalities of the series:

$$-\frac{\pi}{n}+\sum_{j=1}^{+\infty}(-1)^{j+1}\left\{\underbrace{\frac{1}{(2j)!}(\frac{\pi}{n})^{2j}+\frac{1}{(2j+1)!}(\frac{\pi}{n})^{2j+1}\left[(k+1)^{2j+1}-k^{2j+1}\right]}_{a_j}\right\}<0. \tag{17}$$

From $\dfrac{\pi^{2j+3}}{(2j+3)!}<\dfrac{\pi^{2j+1}}{(2j+1)!}$ and $\left[(\frac{k+1}{n})^{2j+1}-(\frac{k}{n})^{2j+1}\right]>\left[(\frac{k+1}{n})^{2j+3}-(\frac{k}{n})^{2j+3}\right]$, for $n>6$; is checked the positive sign of a_j-a_{j+1}, for $j\in\mathbb{N}$, consequently (17) is an alternated series with its first term $-\dfrac{\pi}{n}$ which validates the sign of (17). ■

Notation 3.1.1. Γ_k *symbolizes the euclidean hamiltonian cyclic path determined by the sequence of segments (15), and let Γ_k' represent the non-cyclic euclidean hamiltonian pathways that come from the rooting up of the initial side L_k^- of Γ_k, $1\le k\le\frac{n}{2}-1$.*

Clearly, for each k, the open path Γ_k' is a euclidean hamiltonian non-cyclic path of order $n-1$, with initial and final ending points at V_k and V_0 respectively.

Lemma 3.1.1. *For $n \in \mathbb{N}$ the lengths $l(L_k)$ of the sides $L_k^{\pm}$, $1 \le k \le \lfloor \frac{n}{2} \rfloor$, verify the following relationships:*

$$l(L_{\lfloor \frac{n}{2} \rfloor - i}) - l(L_{\lfloor \frac{n}{2} \rfloor -(i+1)}) < l(L_{\lfloor \frac{n}{2} \rfloor -(i+1)}) - l(L_{\lfloor \frac{n}{2} \rfloor -(i+2)}), \quad 0 \le i \le \lfloor \frac{n}{2} \rfloor - 2. \tag{18}$$

Proof. Since $L_{\lfloor \frac{n}{2} \rfloor - k} = 2\cos\left(\frac{k\pi}{n}\right)$, $0 \le k \le \lfloor \frac{n}{2} \rfloor - 1$, the inequality (18) is equivalent to the following one:

$$\sin\left[\left(\frac{i}{n} + \frac{1}{2n}\right)\pi\right] < \sin\left[\left(\frac{i+1}{n} + \frac{3}{2n}\right)\pi\right],$$

which is true for $0 \le i \le \lfloor \frac{n}{2} \rfloor - 3$. ■

Notation 3.1.2. $\mathfrak{T}'$ symbolizes the family of all the non-cyclic hamiltonian trajectories of order $n-1$ with vertices in the set $\{V_0, \cdots, V_{n-1}\}$ of the $\mathcal{N} = \{K_{n=2p+2}(\sqrt[2p+2]{1}), D\}$'s networks.

Theorem 3.1.4. *In the $\mathcal{N} = \{K_{n=2p+2}(\sqrt[2p+2]{1}), D\}$'s networks, the traversed length of the trajectory Γ_k', $\mathfrak{L}(\Gamma_k')$, $1 \le k \le \frac{n}{2}$, is the maximum of the euclidean hamiltonian non-cyclic path problem of order $n-1$, with initial and final ends at V_k and V_0.*

Proof. The lengths of the non-cyclic, or opened, trajectories Γ_k', verify a reversed relationship to (16) validated by their corresponding cyclic trajectories Γ_k, i.e.,

$$\mathfrak{L}(\Gamma_1') > \mathfrak{L}(\Gamma_2') > \cdots > \mathfrak{L}(\Gamma_{\frac{n}{2}-1}') > \mathfrak{L}(\Gamma_{\frac{n}{2}}'). \tag{19}$$

The core of the present statement relies on the demonstration of the above inequalities.

The maximum numbers of admissible diameters, $l_{\max}$, for any feasible euclidean hamiltonian non-cyclic path of order $n-1$ which touches $n = 2p+2$ nodes of these networks is $\frac{n}{2}$. Therefore, the possible maximum length of a non-cyclic hamiltonian confi guration of order $n-1$ should have $\frac{n}{2}$ diameters $l_{\max}$ and $\frac{n}{2}-1$ quasi-diameters $l_{\text{q.max}}$. Precisely, Γ_1' is a path with such features. That is, $\mathfrak{L}(\Gamma_1') = \max\{\mathfrak{L}(\Gamma') : \Gamma' \in \mathfrak{T}'\}$.

Is there a trajectory $\Gamma' \in \mathfrak{T}'$ such as: $\mathfrak{L}(\Gamma_1') > \mathfrak{L}(\Gamma')$ and $\nexists \overline{\Gamma'} \in \mathfrak{T}' : \mathfrak{L}(\Gamma_1') > \mathfrak{L}(\overline{\Gamma'}) > \mathfrak{L}(\Gamma')$? From *lemma 3.1.1*, the lesser difference between sides L_k is $l_{\max} - l_{\text{q.max}}$. Therefore the minor feasible decreasing of the length Γ_1' is attained if one diameter $l_{\max}$ is replaced by one quasi-diameter $l_{\text{q.max}}$. Then, if such trajectory exists, it should have $\frac{n}{2}-1$ sides $l_{\max}$ and $\frac{n}{2}$ sides $l_{\text{q.max}}$. Exactly, Γ_2' is that kind of trajectory.

By the same argument, if in Γ_2' one side $l_{\max}$ is replaced by $l_{\text{q.max}}$, the resultant trajectory if it exists should have $\frac{n}{2}-2$ side $l_{\max}$ and $\frac{n}{2}+1$ $l_{\text{q.max}}$, as in Γ_3'.

The reiteration of this procedure confi rms the chain of inequalities (19), with the following additional conclusion: any trajectory $\Gamma' \in \mathfrak{T}'$, which its $n-1$ sides are not a reordering of the sides of some trajectory Γ_k', $1 \le k \le \frac{n}{2}$, has the property that $\mathfrak{L}(\Gamma_{\frac{n}{2}}') > \mathfrak{L}(\Gamma')$. Particularly, any euclidean hamiltonian trajectory, non-cyclic, of order $n-1$, with initial and fi nal ends at V_k and V_0, has length less than or equals $\mathfrak{L}(\Gamma_k')$. ■

Theorem 3.1.5. *In $\mathscr{N} = \{K_{n=2p+2}(\sqrt[2p+2]{1}), D\}$'s networks the maxima of the euclidean hamiltonian cyclic problems of order $n = 2p+2$, corresponds to $\mathfrak{L}(\Gamma_{\frac{n}{2}-1})$.*

Proof. For each k, $1 \le k \le \frac{n}{2}$, let $\mathfrak{T}_k$ represent the set of all the trajectories $\Gamma \in \mathfrak{T}$, such that Γ has at least one side L_k (L_k^+ or L_k^-).

Let $\Gamma \in \mathfrak{T}_k$, $1 \le k \le \frac{n}{2} - 1$ and Γ' denote the non-cyclic trajectory obtained if one side L_k is rooted up from Γ. Since *Theorem 3.1.4* it is known that $\mathfrak{L}(\Gamma') \le \mathfrak{L}(\Gamma'_k)$.

From $\mathfrak{L}(\Gamma) = \mathfrak{L}(\Gamma') + l(L_k)$ and $\mathfrak{L}(\Gamma_k) = \mathfrak{L}(\Gamma'_k) + l(L_k)$, it results that $\mathfrak{L}(\Gamma) \le \mathfrak{L}(\Gamma_k)$, for k, $1 \le k \le \frac{n}{2} - 1$. This inequality shows that, for $1 \le k \le \frac{n}{2} - 1$, $\mathfrak{L}(\Gamma_k)$ is the maximum length of any trajectory $\mathfrak{T}$ which has, at least, one side L_k.

The only detail that remains to be seen is to verify if $\Gamma \in \mathfrak{T}_{\frac{n}{2}}$, the inequality $\mathfrak{L}(\Gamma) \le \mathfrak{L}(\Gamma_{\frac{n}{2}-1})$ is true. For this sake, let us consider a trajectory $\Gamma \in \mathfrak{T}_{\frac{n}{2}} - \mathfrak{T}_{\frac{n}{2}-1}$. Hence, Γ does not have quasi-diameters $l_{\text{q.max}}$ and can not have more than $\frac{n}{2}$ diameters $l_{\max}$. In other words, Γ must have $\frac{n}{2}$ sides L_k (L_k^+ or L_k^-) but k only can take values which do not surpass $\frac{n}{2} - 2$, i.e., $\mathfrak{L}(\Gamma) \le \dfrac{n}{2} l_{\max} + \dfrac{n}{2} l(L_{\frac{n}{2}-2})$. Since

$$\frac{n}{2} l_{\max} + \frac{n}{2} l(L_{\frac{n}{2}-2}) = 2\, l_{\max} + (\frac{n}{2} - 2)(l_{\max} - l(L_{\frac{n}{2}-1})) + (\frac{n}{2} - 2)\, l(L_{\frac{n}{2}-1}) + \left(\frac{n}{2} - 2\right) l(L_{\frac{n}{2}-2}) + 2\, l(L_{\frac{n}{2}-2})$$

and from *lemma 3.1.1*,

$$d\frac{n}{2} l_{\max} + \frac{n}{2} l(L_{\frac{n}{2}-2}) < 2\, l_{\max} + (n-4) l(L_{\frac{n}{2}-1}) + 2l(L_{\frac{n}{2}-2}) < \mathfrak{L}(\Gamma_{\frac{n}{2}-1}).$$

Therefore $\mathfrak{L}(\Gamma) < \mathfrak{L}(\Gamma_{\frac{n}{2}-1})$. ■

4. Conclusion

This contribution confirms the optimum configurations of the euclidean hamiltonian cyclic paths of order n in $\mathscr{N} = \{K_n(\sqrt[n]{1}), D\}$'s networks. The resolution of the longest euclidean hamiltonian paths of order $n-1$ in $\mathscr{N} = \{K_{n=2p+2}(\sqrt[2p+2]{1}), D\}$'s networks, paves the way to dealing with other non-cyclic optimum path problems, e.g., the minimum of the euclidean hamiltonian non-cyclic paths of order $n-1$ in the $\mathscr{N} = \{K_n(\sqrt[n]{1}), D\}$'s networks —results just published in [16]— as well as the maximum of the euclidean hamiltonian non-cyclic paths of order $n-1$ in the $\mathscr{N} = \{K_{n=2p+1}(\sqrt[2p+1]{1}), D\}$'s networks —these results have just been proved by us but remain yet unpublished.

Ancillary results of this outcome are the enumeration of the reflective euclidean hamiltonian paths in $\mathscr{N} = \{K_n(\sqrt[n]{1}), D\}$'s networks and the locations of the traversed length of the euclidean hamiltonian cyclic paths that look like the $4p$-stargons of maximum density in $\mathscr{N}(K_{n=4p}(\sqrt[4p]{1}), D)$'s networks, in the sequence of overall lengths of the remarkable suboptimal and optimal cycles.

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APPENDICES

Appendix A: Demonstration of *theorem 2.1.1*. The stationary critic points $\alpha(k)_{c_i\, 1\le i\le n-1}$, $0\le k\le n-1$, must satisfy the trigonometric identities:

$$-\frac{\sin\alpha_1}{\sqrt{1+\cos\alpha_1}}=\frac{\sin(\alpha_1-\alpha_2)}{\sqrt{1-\cos(\alpha_1-\alpha_2)}}=\cdots=\frac{\sin(\alpha_{i-1}-\alpha_i)}{\sqrt{1-\cos(\alpha_{i-1}-\alpha_i)}}=\cdots=\frac{\sin(\alpha_{n-1}-\beta)}{\sqrt{1-\cos(\alpha_{n-1}-\beta)}}, \quad (20)$$

equivalent to the angular differences: $\pi-\alpha_1=\alpha_1-\alpha_2=\alpha_2-\alpha_3=\cdots=\alpha_{i-1}-\alpha_i=\cdots=\alpha_{n-1}-\beta$. Herein, it is understood that $=$ is $\equiv$, where $\delta\equiv\gamma$ if they differ by a multiple of 2π, then a recursive computation brought up equation (2).

Proof. i) Why did each stationary critic point build one reflective trajectory of n-linear branches? Because $\Delta\alpha_{c_i}(k)$ is independent of i. From the fact that the stationary critic points make zero none of the denominators in (20), there are n linear branches which form the paths.

Proof. ii) The critic point $\alpha_{c_i}(1)=(\pi,\cdots,\pi)$ became a singular point which corresponds to the minimum of the function, $F_n(\alpha_{c_i}(1),-\pi)=0$.

Proof. iii) For brevity's sake, let $a_1=\sqrt{1+\cos\alpha_1}$, $a_j=\sqrt{1-\cos(\alpha_{j-1}-\alpha_j)}$ for $2\le j\le n-1$, $a_n=\sqrt{1-\cos(\alpha_{n-1}-\beta)}$ and let a^i_j's stand for $a^i_j=a_j$ if $j\neq i$ meanwhile $a^i_j=1$ if $j=i$. The evaluations of $F_{\alpha_1\alpha_1}=-\frac{\sqrt{2}R}{4}\{a_1+a_2\}$ and the determinants of the Hessian matrices, $\mathscr{H}_i$, i.e., $|\mathscr{H}_i|=(-\frac{\sqrt{2}R}{4})^i\sum_{i=1}^{i+1}\prod_{j=1}^{i+1}a^i_j$, if $2\le i\le n-2$, and $|\mathscr{H}_{n-1}|=(-\frac{\sqrt{2}R}{4})^{n-1}\sum_{i=1}^{n}\prod_{j=1}^{n}a^i_j$ confirm their alternate sign [11], therefore the function (1) has relative maxima at whichever set of stationary critic points.

Proof. iv) Why did each singular critic point build reflective paths with less than n linear branches? This outcome comes from the very definition of the singular critic points. It means that at least one of the denominators in (20) must be zero, consequently at least one of the square roots in (1) will have null contribution.

Proof. v) The $\min(F_n(\alpha_i,\beta))=\sqrt{2}\sqrt{1+\cos(\beta)}=F_n(\pi,\cdots,\pi,\beta)$, is attained at the singular point $(\pi,\cdots,\pi,\beta)$.

Proof. vi) Follows from lemmas 4.0.2 and 4.0.3.

Lemma 4.0.2. *For $n\ge 5$ and each $\beta\in[-\pi,0]$ the longest of the relative maxima of $F_n(\alpha_i,\beta))$ is longer than any path attained at the singular critical points.*

Proof. $\bar{\alpha}_i$ is a singular point if and only if at least one of the n addend of $F_n(\bar{\alpha}_i,\beta)$ is null. Hence $F_n(\bar{\alpha}_i,\beta)\le 2(n-1)$, therefore the statement is validated if the following inequalities are true:

$$i)\quad \text{If } n \text{ odd;}\quad 2(n-1)<n\sqrt{2}\sqrt{1+\cos(\frac{\beta}{n})}=F_n(\alpha_{c_i}(\frac{n+1}{2}),\beta) \qquad (21)$$

$$ii)\text{If } n \text{ even;}\quad 2(n-1)<n\sqrt{2}\sqrt{1+\cos(\frac{\beta+\pi}{n})}=F_n(\alpha_{c_i}(\frac{n}{2}+1),\beta) \qquad (22)$$

Firstly, we prove $i)$: $2(n-1)<n\sqrt{2}\sqrt{1+\cos(\frac{\beta}{n})}\ \Leftrightarrow\ 2(1-\frac{1}{n})^2<1+\cos(\frac{\beta}{n})\ \Leftrightarrow\ -4+\frac{2}{n}<\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\frac{\beta^{2k}}{n^{2k-1}}$. The series in the right hand term converges to S, with $S>-\frac{\beta^2}{2n}$. The

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last inequality becomes true if $\frac{\beta^2}{2n} \leq 4 - \frac{2}{n}$. Hence (21) is validated, since $\frac{\beta^2}{2n} \leq \frac{\pi^2}{2n} < \frac{5}{n} < 4 - \frac{2}{n}$, $\forall\, n \geq 2$. Secondly, we verify *ii*): Analogous arguments confirm the inequality (22) from $\frac{(\beta+\pi)^2}{2n} \leq \frac{\pi^2}{2n} < 4 - \frac{2}{n}$, $\forall\, n \geq 2$. □

Lemma 4.0.3. *For each $\beta \in [-\pi, 0]$ the maximum of $F_n(\alpha_i, \beta)$ is a relative maximum.*
Proof. For each $\beta \in [-\pi, 0]$ the $n-1$ angles $\alpha_i \in \mathbb{R}$ (see Figure 1) are selected as independent variables of $F_n(\alpha_i, \beta)$. $F_n(\alpha_i, \beta)$ is a continuous function everywhere. Furthermore $F_n(\alpha_i, \beta) = F_n(\alpha_i + 2k_i\pi, \beta)$ for any k_i integer, hence $F_n(\alpha_i, \beta)$ attains all the values of its image, over the compact set $\mathscr{K} = [0, 2\pi]^{n-1} \times [-\pi, 0]$.

If β_* is a particular selection from $[-\pi, 0]$, the function $F_n(\alpha_i, \beta_*)$ attains its maximum at (α_i^*, β_*) in the compact set $\mathscr{K}_* = [0, 2\pi]^{n-1} \times \{\beta_*\}$.
By *lemma 4.0.2*, $F_n(\alpha_i, \beta_*)$ is a differentiable function at (α_i^*, β_*), therefore such maximum is a relative maximum, the longest of the relative maxima of $F_n(\alpha_i, \beta_*)$. □
Proof. vii) and viii) For $-\pi < \beta < 0$,

$$\Delta\alpha_{ci}(k) = \frac{\pi - \beta}{n} - k\frac{2\pi}{n}. \tag{23}$$

The following chains of inequalities come from (23) and prove (3) and (6), respectively.

$$\begin{aligned} &0 < -\Delta\alpha_{c_i}(1) < \Delta\alpha_{c_i}(0) < -\Delta\alpha_{c_i}(2) < \Delta\alpha_{c_i}(n-1) + 2\pi < -\Delta\alpha_{c_i}(3) < \\ &< \underbrace{\Delta\alpha_{c_i}(n-j) + 2\pi < -\Delta\alpha_{c_i}(j+2)}_{2 \leq j \leq \frac{n}{2}-2} < \Delta\alpha_{c_i}(\frac{n}{2}+1) + 2\pi < \pi. \end{aligned} \tag{24}$$

$$0 < -\Delta\alpha_{c_i}(1) < \Delta\alpha_{c_i}(0) < -\Delta\alpha_{c_i}(2) < \underbrace{\Delta\alpha_{c_i}(n-j) + 2\pi < -\Delta\alpha_{c_i}(j+2)}_{1 \leq j \leq \frac{n-3}{2}} < \pi. \tag{25}$$

For $\beta = -\pi$, since $-\Delta\alpha_{ci}(k) = (k-2)\frac{2\pi}{n}$, (26) and (27) are obtained, which prove (4) and (7), respectively.

$$\begin{aligned} &0 = -\Delta\alpha_{c_i}(1) < \Delta\alpha_{c_i}(0) = -\Delta\alpha_{c_i}(2) < \Delta\alpha_{c_i}(n-1) + 2\pi = -\Delta\alpha_{c_i}(3) < \\ &< \underbrace{\Delta\alpha_{c_i}(n-j) + 2\pi = -\Delta\alpha_{c_i}(j+2)}_{2 \leq j \leq \frac{n}{2}-2} < \Delta\alpha_{c_i}(\frac{n}{2}+1) + 2\pi = \pi. \end{aligned} \tag{26}$$

$$0 = -\Delta\alpha_{c_i}(1) < \Delta\alpha_{c_i}(0) = -\Delta\alpha_{c_i}(2) < \underbrace{\Delta\alpha_{c_i}(n-j) + 2\pi = -\Delta\alpha_{c_i}(j+2)}_{1 \leq j \leq \frac{n-3}{2}} < \pi. \tag{27}$$

For $\beta = 0$, since $-\Delta\alpha_{ci}(k) = (2k-1)\frac{2\pi}{n}$, (28) and (29) are obtained, which prove (5) and (8), respectively.

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$$0 < -\Delta\alpha_{c_i}(1) = \Delta\alpha_{c_i}(0) < -\Delta\alpha_{c_i}(2) = \Delta\alpha_{c_i}(n-1) + 2\pi < \\ < \underbrace{-\Delta\alpha_{c_i}(j) = \Delta\alpha_{c_i}(n+1-j) + 2\pi}_{3\leq j\leq \frac{n}{2}} < \pi. \tag{28}$$

$$0 < -\Delta\alpha_{c_i}(1) = \Delta\alpha_{c_i}(0) < \underbrace{-\Delta\alpha_{c_i}(j) = \Delta\alpha_{c_i}(n+1-j) + 2\pi}_{2\leq j\leq \frac{n-1}{2}} < -\Delta\alpha_{c_i}(\frac{n+1}{2}) = \pi. \tag{29}$$

Observation 4.0.1. *If k and $\tilde{k}$ label two distinct reflective paths, with respective angular values $\Delta\alpha_{c_i}(k)$ and $\Delta\alpha_{c_i}(\tilde{k})$, both reflective paths have the same length if and only if $\Delta\alpha_{c_i}(k) \equiv \pm\Delta\alpha_{c_i}(\tilde{k})$. However this equality is possible if and only if $\beta = 0$ or $\beta = -\pi$, with $\tilde{k} = n+1-k$ and $\tilde{k} = n+2-k$, respectively. In conclusion, $\forall \beta \in (-\pi, 0)$ all the reflective paths have distinct lengths. On the other hand, if $\beta = 0$ is $F_n(\alpha_{c_i}(k), 0) = F_n(\alpha_{c_i}(n+1-k), 0)$ and if $\beta = -\pi$ is $F_n(\alpha_{c_i}(k), -\pi) = F_n(\alpha_{c_i}(n+2-k), -\pi)$, these cycles were object of special attention in §2.1.1 and §2.1.2. The case $\beta = 0$ is explained in* [17].

Proof. ix) The reflective paths when $-\pi < \beta < 0$ travels in ccw. or cw. circulation, according with $\pi < \alpha_{c_1}(k) < 2\pi$ or $2\pi < \alpha_{c_1}(k) < 3\pi$, respectively.

Direct calculations show that the scheme in (9) is true. In $\beta = -\pi$ and $\beta = 0$ the paths determined by $k = \frac{n}{2} + 1$ and $k = \frac{n+1}{2}$ render, respectively $\alpha_{c_1}(k) = 2\pi$. Therein we assigned the ccw. circulation. Then, simple computations drive to (10) and (11). ∎

Appendix B: Demonstration of *theorem 3.1.2*. Sequence (15) determines the ordering of the sides, it starts up with L_k^- that links V_0 to V_k and ends up with the latest $l_{\max}$ which joins certain vertex to V_0. The vertices $V_0, \cdots, V_{n-1}$ are renumbered in the following manner: $V_0' = V_0$ and if $1 \leq j \leq n-1$, V_j' designates the vertex that determines the j-th side of the sequence (15) when it is located after V_{j-1}'. That is, $V_0' = V_0$, $V_1' = V_k$, $V_2' = V_{k+\frac{n}{2}-1}$, $\cdots$. Let $\sphericalangle(V_j')$ symbolize the angle of c.w. advance, corresponding to the vertex V_j', the angular sequence in correspondence to the vertices $V_0', \cdots, V_{n-1}'$ starts with the following values:

$$\sphericalangle(V_0') = \pi,\ \sphericalangle(V_1') = \pi - k\frac{2\pi}{n},\ \sphericalangle(V_2') = (-k+1)\frac{2\pi}{n},\ \sphericalangle(V_3') = (-k+2)\frac{2\pi}{n} - \pi, \cdots$$

There is a natural partition of the sequence (15) in three sections S_1, S_2 and S_3. The first, S_1, made up of a single L_k^- in first place and of $k-1$ sides $l_{\text{q.max}}^-$. The second, S_2, made up of one initial diameter $l_{\max}$ and of k quasi-diameters $l_{\text{q.max}}^+$. Finally, the third S_3, made up of $\left(\frac{n}{2} - k\right)$ diameters $l_{\max}$ and $\frac{n}{2} - k - 1$ quasi-diameters $l_{\text{q.max}}^+$. They appear alternated and the first and the last are diameters $l_{\max}$.

Each section can be characterized by its associated vertices, that is:

$$S_1 = \{V_0', \cdots, V_k'\}, \quad S_2 = \{V_{k+1}', \cdots, V_{2k+1}'\}, \quad \text{and} \quad S_3 = \{V_{2k+2}', \cdots, V_{2k+(n-2k)}' = V_n'\}.$$

The sequences of vertices and angles determined by the sequence of segments (15), split by sections, is schematized below:

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$$S_1: \quad \underset{V_0'}{\pi} \xrightarrow{L_k^-} \underset{V_1'}{\left[\pi - k\frac{2\pi}{n}\right]} \cdots \xrightarrow{l^-_{q.max}} \cdots \underset{V_j'}{\left[\pi - k\frac{2\pi}{n} - (j-1)\left(\pi - \frac{2\pi}{n}\right)\right]} \cdots \xrightarrow{l^-_{q.max}} \cdots \underset{V_k'}{\left[-(k-2)\pi - \frac{2\pi}{n}\right]}$$

$$S_2: \quad \xrightarrow{l_{\max}} \underset{V_{k+1}'}{\left[-(k-1)\pi - \frac{2\pi}{n}\right]} \cdots \xrightarrow{l^+_{q.max}} \cdots \underset{V_{k+i}'}{\left[-(k+i-2)\pi - i\frac{2\pi}{n}\right]} \cdots \xrightarrow{l^+_{q.max}} \cdots \underset{V_{2k+1}'}{\left[-(2k-1)\pi - (k+1)\frac{2\pi}{n}\right]}$$

$$S_3: \quad \xrightarrow{l_{\max}} \underset{V_{2k+2}'}{\left[-2k\pi - (k+1)\frac{2\pi}{n}\right]} \xrightarrow{l^+_{q.max}} \cdots \quad \overset{\overbrace{l_{\max}}^{\text{if } j=2s}}{\underset{\underbrace{l^+_{q.max}}_{\text{if } j=2s+1}}{\rightarrow}} \quad \underset{\substack{V_{2k+j}' \\ (2\le j\le n-2k)}}{\left\{ \begin{array}{l} \left[-(2k+j-2)\pi - (k+s)\frac{2\pi}{n}\right] \\ \\ \left[-(2k+j-2)\pi - (k+s+1)\frac{2\pi}{n}\right] \end{array} \right.} \quad \rightarrow \cdots$$

$$\xrightarrow{l_{\max}} \underset{V'_{2k+(n-2k)=V'_n\equiv V'_0=V_0}}{[-(n+1)\pi]}$$

The demonstration is attained in two stages, in the first one —a_1, a_2, a_3— it is checked that each section S_i is made up of different vertices, and in the second one —b_1, b_2, b_3— it is confirmed that $S_i \cap S_j = \emptyset$ if $i \neq j$.

$a_1)$ On account of the way in which the subsequent vertices are obtained from V_0', in S_1, any two vertices $V_i' \neq V_j'$, $1 \le i < j \le k$, if the angular difference $\sphericalangle(V_i') - \sphericalangle(V_j') = (j-i)\pi + (i-j)\frac{2\pi}{n} \neq 2m\pi$, $m \in \mathbb{Z}^-$. This comes from $0 < \frac{2(j-i)}{n} \le 1 - \frac{6}{n}$.

$a_2)$ Similar to the case (a_1), in S_2, any two vertices $V_{k+i}' \neq V_{k+j}'$, $1 \le i < j < k+1$, requires that $\sphericalangle(V_{k+i}') - \sphericalangle(V_{k+j}') = (j-i)\pi + (j-i)\frac{2\pi}{n} \neq 2m\pi$, $m \in \mathbb{Z}^-$. It comes since $\frac{2(j-i)}{n} \notin \mathbb{Z}^-$.

$a_3)$ In S_3, the expressions of the angles of the vertices differ from even or odd order of the vertex. Therefore, the differences $\sphericalangle(V_{2k+i}') - \sphericalangle(V_{2k+j}')$ for $2 \le i < j \le n-2k$, should be evaluated for i even with j even, i even with j odd, i odd with j even and i odd with j odd:

- $i < j$, (i even, j even): $\sphericalangle(V_{2k+i}') - \sphericalangle(V_{2k+j}') = (j-i)\pi + \left(\frac{j-i}{n}\right)\pi$. Since $\frac{j-i}{n} \notin \mathbb{Z}$, the vertices do not coincide.

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- $i < j$, (i even, j odd): $\sphericalangle(V'_{2k+i}) - \sphericalangle(V'_{2k+j}) = (j-i)\pi + \left(\frac{j-1-i}{n}\right)\pi$. Since $\frac{j-i-1}{n} \notin \mathbb{Z}$, the vertices do not coincide.
- $i < j$, (i odd, j even): $\sphericalangle(V'_{2k+i}) - \sphericalangle(V'_{2k+j}) = (j-i)\pi + (j-i-1)\frac{\pi}{n}$. From $\frac{j-i-1}{n} \notin \mathbb{Z}$, the vertices do not coincide.
- $i < j$, (i odd, j odd): $\sphericalangle(V_{2k+i}) - \sphericalangle(V_{2k+j}) = (j-i)\pi + (j-i)\frac{\pi}{n}$. Since $\frac{j-i}{n} \notin \mathbb{Z}$, the vertices do not coincide.

The empty intersection between the three sections requires the same kind of procedure:

b_1) $S_1 \cap S_2 = \emptyset$. It is sufficient to prove that $\sphericalangle(V'_j) - \sphericalangle(V'_{k+i}) = (k+i-j)\pi + (i+j-k-1)\frac{2\pi}{n} \neq 2m\pi, m \in \mathbb{Z}^-$ if $1 \leq i \leq k+1$; $1 \leq j \leq k \leq \frac{n}{2}-2$. Since $-1+\frac{2}{n} \leq \frac{2(i+j-k-1)}{n} \leq 1-\frac{2}{n}$, even if $(i+j-k-1)=0 \Rightarrow \sphericalangle(V'_j) - \sphericalangle(V'_{k+i}) = (2i-1)\pi \neq 2m\pi$.

b_2) $S_2 \cap S_3 = \emptyset$. It is necessary to prove that $\sphericalangle(V'_{k+i}) - \sphericalangle(V'_{2k+j}) \neq 2m\pi, m \in \mathbb{Z}^-$, if $1 \leq i \leq k+1, 2 \leq j \leq n-2k, 1 \leq k \leq \frac{n}{2}-2$ when j takes even or odd values.

If j is even, $\sphericalangle(V'_{k+i}) - \sphericalangle(V'_{2k+j}) = (k-i+j)\pi + \left(k+\frac{j}{2}-i\right)\frac{2\pi}{n}$. But $\frac{2(k+j/2-i)}{n} \in \mathbb{Z}$, if and only if it is zero. Then $\sphericalangle(V'_{k+i}) - \sphericalangle(V'_{2k+j}) = \pi \neq 2m\pi, m \in \mathbb{Z}^-$.

If j is odd, j only can vary between 3 and $n-2k-1$, because the latest $j = n-2k$ is even. In such a situation we will see $\sphericalangle(V'_{k+i}) - \sphericalangle(V'_{2k+j}) = (k+j-i)\pi + (k+j/2-i+1)\frac{2\pi}{n} \neq 2m\pi, m \in \mathbb{Z}^-$. If $1 \leq k \leq \frac{n}{2}-2$, $1 \leq i \leq k+1$ and $j = 2s+1$ is such that $3 \leq j \leq n-2k-1$, then $j = 2$ and $j = n-2k$ are even. Here $1 \leq s \leq \frac{n}{2}-(k+1)$. Therefore $\sphericalangle(V'_{k+i}) - \sphericalangle(V'_{2k+j}) = (k+j-i)\pi + (k+s-i+1)\frac{2\pi}{n}$. Since $1 \leq s \leq \frac{n}{2}-k-1$ it results that $\frac{6-2i}{n} \leq \frac{2(s+k-i+1)}{n} < 1$, then $\frac{2(s+k-i+1)}{n}$ only can be zero or a non integer fraction. If it were zero, $s = i-k-1$ should be less than or equals zero, it contradicts $s \geq 1$.

b_3) Finally, to demonstrate that $S_1 \cap S_3 = \emptyset$ it is needed to prove, that $\sphericalangle(V'_i) - \sphericalangle(V'_{2k+j}) \neq 2m\pi, m \in \mathbb{Z}^-$, if $1 \leq k \leq \frac{n}{2}-2$, $1 \leq i \leq k$ and $2 \leq j \leq n-2k$, dealing with the cases j even and j odd separately.

If j is even, $j = 2s$ with $1 \leq s \leq \frac{n}{2}-k$ and $\sphericalangle(V'_i) - \sphericalangle(V'_{2k+j}) = (2k+j-i)\pi + (i+s-1)\frac{2\pi}{n} \neq 2m\pi, m \in \mathbb{Z}^-$ since $0 < \frac{(i+s-1)2}{n} < 1$.

If j is odd, it should be $3 \leq j = 2s+1 \leq n-2k-1$, since 2 and $n-2k$ are even. Hence $1 \leq s \leq \frac{n}{2}-k-1$ and then $\sphericalangle(V'_i) - \sphericalangle(V'_{2k+j}) = (2k+j-i)\pi + (i+s)\frac{2\pi}{n}$, with $\frac{(i+s)2}{n} \notin \mathbb{Z}$ because $\frac{4}{n} \leq \frac{2(s+i)}{n} \leq 1-\frac{2}{n}$. ∎

Appendix C: Hamiltonian paths that look like the stargons of maximum density in $\mathcal{N}(K_{4p}(\sqrt[4p]{1}), D)$'s networks.

Theorem 4.0.6. *In $\mathcal{N} = \{K_{4p}(\sqrt[4p]{1}), D\}$'s networks with $p \geq 2$, the length of the euclidean hamiltonian cyclic paths $T_\bigstar$, that look like the $4p$-stargons of maximum density, is located between the terms corresponding to $k = \frac{n}{2}-3$ and $k = \frac{n}{2}-2$ in the sequence (16) of lengths of paths Γ_k, i.e.,*

$$\mathfrak{L}(\Gamma_1) < \mathfrak{L}(\Gamma_2) < \cdots < \mathfrak{L}(\Gamma_{\frac{n}{2}-3}) < \mathfrak{L}(T_\bigstar) < \mathfrak{L}(\Gamma_{\frac{n}{2}-2}) < \mathfrak{L}(\Gamma_{\frac{n}{2}-1}). \tag{30}$$

Proof. It is necessary to validate the inequalities below:

$$4\,l_{\max} + (n-5)\,l_{\text{q.max}} + l_{\max}\cos(\frac{3\pi}{n}) < n\,l_{\text{q.max}}, \tag{31}$$

$$3\,l_{\max} + (n-4)\,l_{\text{q.max}} + l_{\max}\cos(\frac{2\pi}{n}) > n\,l_{\text{q.max}}. \tag{32}$$

Inequality (31) turns into $\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{3\pi}{n}\right)^{2k}\left[-\frac{5}{3^{2k}}+1\right] < 0.$

Let $a_k = \frac{1}{(2k)!}(\frac{3\pi}{n})^{2k}\left[-\frac{5}{3^{2k}}+1\right]$ and since $a_k - a_{k+1} > 0,$ the series is alternated, its first term, $a_1 = S_1 = -\frac{9}{2}(\frac{\pi}{n})^2\,[-\frac{5}{9}+1] < 0,$ is negative, therefore the sum of the series is negative. Finally, (31) is true.

The inequality (32) is true, it stands for $2\sin^2\left(\frac{\pi}{n}\right)\left(\frac{2}{1+\cos(\frac{\pi}{n})}-1\right) > 0, \quad \forall n.$ ∎

The following sketch represents the suboptimal and optimal hamiltonian cycles in the $\mathcal{N} = \{K_{n=12}(\sqrt[12]{1}), D\}$'s networks.

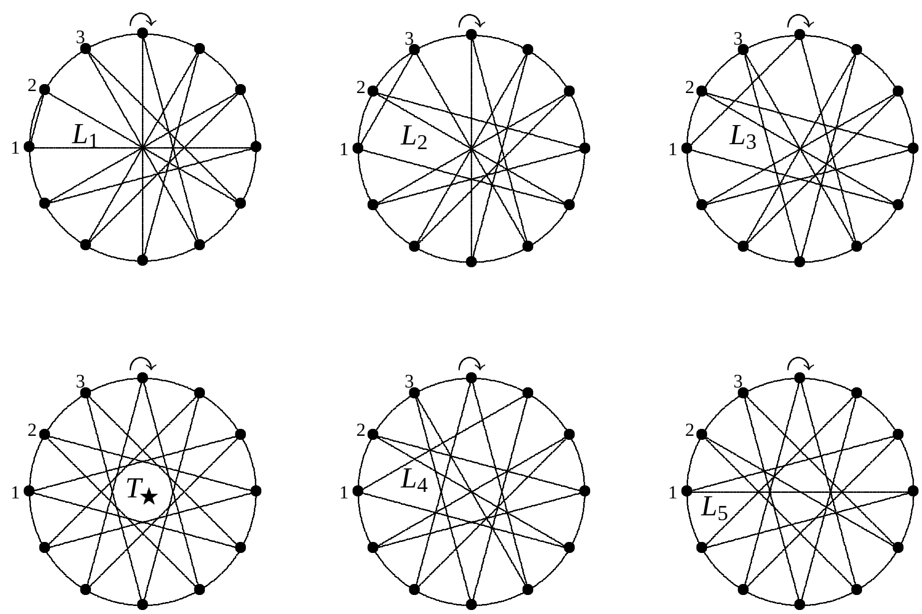

FIGURE 3. From the anti-greedy suboptimal (Γ_1) to the longest (Γ_5) hamiltonian cycles in the $\mathcal{N}(K_{12}(\sqrt[12]{1}), (d_{ij})_{12\times 12})$'s network

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