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AN INVERSE PROBLEM FOR INTEGRABLE SYSTEMS LINEARIZING ON ELLIPTIC CURVES

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ABSTRACT. We obtain algebraic completely integrable systems related to the geometric data given by elliptic curves, configuration divisors on the curves, and the action of a group on the theta sections of the line bundles associated to these divisors . A description of all meaningful systems and Poisson manifolds for the above data with divisors consisting of three points and group of symmetries S_3 is given.

1. INTRODUCTION

Since Euler, interesting connections between finite dimensional integrable systems and algebraic geometry were found.

Most of the known examples of integrable systems are a particular class of completely integrable systems, whose solutions, expressible in terms of theta functions, are associated with abelian varieties and divisors on them, so that the complex hamiltonian flows are linear on these abelian varieties. Roughly speaking, such systems are called algebraic completely integrable (a.c.i.)[1].

Starting from an a.c.i. system we can produce explicit geometric data like a divisor on an abelian variety (the divisor at infinity), its polarization , the linear system associated with this divisor, and a finite group of translations leaving invariant the divisor and the holomorphic vector fields.

In this paper we present an inverse problem and show how to obtain "mathematical" algebraic completely integrable systems from geometric data.

We start with a family of elliptic curves E_α; their configuration divisors at infinity $\mathcal{D}_\alpha$, which in these examples will be the sum of three points; a group $G = \mathbb{Z}_3 \ltimes \mathbb{Z}_2 = S_3$ of symmetries leaving invariant each $\mathcal{D}_\alpha$ (essentially they are translates by third periods and reflection about the origin in the elliptic curve); and line bundles $\mathrm{L}_\alpha \to \mathrm{E}_\alpha$, whose sections are projective coordinates of the ambient space $\mathbb{P}^3$. We provide a construction of the systems by finding a convenient basis $\{U, V, W\}$ of theta functions for the above data, with the property that the same theta functions (up to permissible change of basis) become sections of the line bundles $\mathrm{L}_\alpha \to \mathrm{E}_\alpha$. On these sections, G (a subgroup of the Theta group associated with $\mathcal{D}_\alpha$) acts via the usual Schrödinger representation induced by the Theta group, which is the same for all elliptic curves. We deduce a linear and a cubic equation for the image of the

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curves in $\mathbb{P}((H^0(\mathrm{E}_\alpha, \mathrm{L}_\alpha) \oplus \mathbb{C})^*) = \mathbb{P}^3$, in terms of parameters. Also, we find the quadratic equations for the holomorphic vector fields in terms of the above basis.

The equations that describe the image of E_α in $\mathbb{P}^3$ contain parameters α which serve as a kind of moduli data. Now, one of the theta sections, say Z, will cut out on each E_α a divisor at infinity of the form $\mathcal{D} = e_0 + e_1 + e_2$, and in the affine variables $\frac{\mathrm{U}}{\mathrm{Z}}$, $\frac{\mathrm{V}}{\mathrm{Z}}$, $\frac{\mathrm{W}}{\mathrm{Z}}$ of $\mathbb{C}^3$ we obtain a smooth affine elliptic curve for each generic α . The question is whether such a family of affine curves put together in $\mathbb{C}^3$ has a Poisson structure so that they are the complex invariant manifolds for a hamiltonian structure. We take affine Poisson structures of the form $\{f, g\} =< \mathrm{grad} f, \mathrm{J}.\mathrm{grad} g >$, where J is a Skew-symmetric matrix with polynomial entries in the affine variables.

This leads us to the following theorem that will be shown along the paper.

Theorem 1. *Consider the family $\{\mathrm{E}_\alpha\}$ of elliptic curves and divisors $\{\mathcal{D}_\alpha = e_0 + e_1 + e_2\}$ on them, such that $\mathcal{D}_\alpha$ is invariant under a translate by a $\frac{1}{3}$-period, and $3e_0 \sim_\ell 3e_1 \sim_\ell 3e_2 \sim_\ell \mathcal{D}_\alpha$ ($\sim_\ell$ is linear equivalence of divisors). This family posses a group $G = \mathbb{Z}_3 \ltimes \mathbb{Z}_2 = S_3$ leaving invariant each $\mathcal{D}_\alpha$ and $\mathrm{E}_\alpha \backslash \mathcal{D}_\alpha$. Let $H^0(\mathrm{E}_\alpha, \mathcal{D}_\alpha)$ be the space of sections linearly equivalent to $\mathcal{D}_\alpha$. There is a basis $\{1, u = \frac{\mathrm{U}}{\mathrm{Z}}, v = \frac{\mathrm{V}}{\mathrm{Z}}, w = \frac{\mathrm{W}}{\mathrm{Z}}\}$ for the augmented space $H^0(\mathrm{E}_\alpha, L_\alpha) \oplus \mathbb{C}$, with Z the section vanishing on $\mathcal{D}_\alpha$, such that G acts as in the following table, for all generic curves E_α:*

	u	v	w
σ	v	w	u
ι	u	w	v

The image of E_α in $\mathbb{P}^3 = \mathbb{P}((H^0(\mathrm{E}_\alpha, \mathcal{D}_\alpha) \oplus \mathbb{C})^*)$ is the complete intersection of a linear and a cubic invariants under the group G. The G-invariant polynomial Poisson structures J in the affine variables u, v, w's are quadratic for a system to linearize on the elliptic curves. Then, there is a five-parameter family of nonequivalent integrable systems with nontrivial hamiltonians given by the linear invariant, quadratic matrices J, and cubic Casimir. Also, there is a case of constant matrix J with linear invariant as Casimir and cubic hamiltonian. This gives a four-parameter family of integrable systems with nonequivalent systems characterized by three effective parameters. Here, equivalence means a map from $\mathbb{C}^3 \times \{space\ of\ parameters\}$ into itself, such that the Poisson structure is preserved, as well as the action of the group G, the curves, divisors and holomorphic vector fields. Altogether, we get a three-parameter family of Poisson manifolds.

The method of the proof is quite similar (with modifications adapted to 1-dimensional cases) to that in [7] and uses results of [6]. I want to thank Pol Vanhaecke for suggesting this problem.

2. Family of elliptic curves with a symmetric divisor $\mathcal{D} = e_0 + e_1 + e_2$ at infinity

Let us start with an elliptic curve E= $\mathbb{C}/ < 1, \tau >$ in which an origin e_0 is fixed and two $\frac{1}{3}$-period points $e_1, e_2 = e_1 + e_1$ are given. We consider the divisor $\mathcal{D} = e_0 + e_1 + e_2$, which is invariant under the permutations' group $S_3 =< \sigma, \iota >$

with relations $\sigma^2\iota = \iota\sigma$, $\iota^2 = 1$, $\sigma^3 = 1$. Here σ corresponds to a translation by a $\frac{1}{3}$-period and ι to the (-1) involution. $\mathcal{D}$ can be obtained from a single point (chosen as origin), by applying the group $\mathbb{Z}/3$ generated by a translation by a $\frac{1}{3}$-period. However, we put the condition that $3e_0$, $3e_1$, $3e_2$ are linearly equivalent $\mathcal{D}$, and that imposes restrictions on the possibilities for our divisor $\mathcal{D}$. Namely, not any triple of points $\{e_0, e_1, e_2\}$, subgroup of $\frac{1}{3}$-periods can be considered as $\mathcal{D}$. Let us find now a basis for $H^0(E, \mathcal{O}(\mathcal{D}))$. Since the divisors $3e_0$, $3e_1$, $3e_2$, are also linearly equivalent to $\mathcal{D}$, we take a basis of sections of $\mathcal{O}(\mathcal{D})$ that vanish three times at e_0, three times at e_1, and three times at e_2. Call its elements, respectively U, V, and W. Let also Z be a section that vanishes only once at $\mathcal{D}$. By rescaling, we can pick Z such that σZ=Z and ιZ=Z. Also, by conveniently rescaling U, V, and W, we obtain a basis in which the action of S_3 is given by the following table:

(1)

	Z	U	V	W
σ	Z	V	W	U
ι	Z	U	W	V

To show this, we consider theta functions with characteristics (m, m^*). Let τ be a complex number with positive imaginary part. The pair of real numbers (m, m^*) is associated univocally with the point $m^* + m\tau$ of $\mathbb{C}$.

We define the classical elliptic theta functions with characteristics (m, m^*) [2, §8.5] [5] as (1) below, where $e(z) = exp(2\pi iz)$, $z \in \mathbb{C}$. They have the properties (2),(3),(3'),(4).

(1) $\vartheta_{m,m^*}(\tau, \zeta) = \sum_{n\in\mathbb{Z}} e(\frac{1}{2}(n+m)^2\tau + (n+m)(\zeta + m^*))$

(2) $\vartheta_{m,m^*}(\tau, -\zeta) = \vartheta_{-m,-m^*}(\tau, \zeta)$

(3) $\vartheta_{m+n,m^*+n^*}(\tau, \zeta) = e(mn^*)\vartheta_{m,m^*}(\tau, \zeta)$, for $n, n^* \in \mathbb{Z}$

(3') $\vartheta_{m,m^*}(\tau, \zeta + u\tau + u^*) = e(-\frac{1}{2}u^2\tau - u(\zeta + u^*))e(-um^*)\vartheta_{m+u,m^*+u^*}(\tau, \zeta)$.

Let $(m, m^*) \in \frac{1}{3}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ be a third period, then we have the formula .

(4) $\vartheta_{m,m^*}(\tau, -\zeta) = e(-m)\vartheta_{1-m,1-m^*}(\tau, \zeta)$.

So, we fix $e_0 = 0$, $e_1 = m^* + m\tau$, and $e_2 = 2m^* + 2m\tau$ as representing our group of third periods and we want to see how $\sigma = t_{e_1}$ and $\iota = (-1)$ act on the functions ϑ, ϑ_{m,m^*}, $\vartheta_{2m,2m^*}$, with the action defined by $t_x f(y) = f(y + x)$ and $\iota f(y) = f(-y)$. This is drawn in the following table:

(2)

	$s_1 = \vartheta$	$s_2 = \vartheta_{m,m^*}$	$s_3 = \vartheta_{2m,2m^*}$
σ	$f(\zeta)\vartheta_{m,m^*}$	$f(\zeta)e(-mm^*)\vartheta$	$f(\zeta)e(-2mm^*)\vartheta_{m,m^*}$
ι	ϑ	$e(m(1-6m^*))\vartheta_{2m,2m^*}$	$e(-m(1+6m^*))\vartheta_{m,m^*}$

Let $u_i = s_i^3$, $i = 1 \ldots 3$, and $\Theta = s_1 s_2 s_3$. With the assumption $3e_0 \sim_\ell 3e_1 \sim_\ell 3e_2$, we deduce that $e(-mm^*)$ has to be a cubic root of unity (this will be our restriction to $\mathcal{D}$ since $e(-mm^*)$ is a ninth root of 1). We obtain:

(3)

	u_1	u_2	u_3	Θ
σ	$f^3 u_2$	$f^3 u_3$	$f^3 u_1$	$f^3\Theta$
ι	u_1	u_3	u_2	Θ

Now, the functions $\mathrm{U}=\frac{u_1}{\Theta}$, $\mathrm{V}=\frac{u_2}{\Theta}$, $\mathrm{W}= \frac{u_3}{\Theta}$, and Z=1, thought of as sections in $\mathrm{H}^0(\mathrm{E}, \mathcal{O}(\mathcal{D}))$ give precisely table (1).

Using Riemann-Roch's Theorem for curves [3], we deduce that $\dim \mathrm{H}^0(\mathrm{E}, \mathcal{O}(\mathcal{D})) = 3$. So, Z is a linear combination of $\{\mathrm{U}, \mathrm{V}, \mathrm{W}\}=$ basis for $\mathrm{H}^0(\mathrm{E}, \mathcal{O}(\mathcal{D}))$. To check that this set of sections is linearly independent, one uses the expressions of s_1^3, s_2^3, s_3^3 evaluated at the points e_0, e_1, e_3 above, to obtain the nonsingular matrix $(s_i^3(e_j))$. The relation among U,V,W, and Z must be one of the equations defining the elliptic curve in $\mathbb{P}^3 = \mathbb{P}(<\mathrm{Z}> \oplus <\mathrm{U}, \mathrm{V}, \mathrm{W}>)$. Thus, we get the linear relation

$$\mathrm{U} + \mathrm{V} + \mathrm{W} = a\mathrm{Z} \tag{4}$$

since it has to be invariant under the action of the group (1).

We consider the four dimensional space $\mathrm{H} =<\mathrm{Z}> \oplus \mathrm{H}^0(\mathrm{E}, \mathcal{O}(\mathcal{D}))$. There is an exact sequence of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & \mathcal{O}_{\mathrm{E}}(1) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathcal{O}_{\mathrm{E}}(\mathcal{D}) & & \end{array}$$

that induces an exact cohomology sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathcal{I}(1)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(1)) & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & \mathrm{H} & & & & \\ & & \dim = 1 & & \dim = 4 & & \dim = 3 & & \end{array}$$

since we have only one linear equation defining the elliptic curve E in $\mathbb{P}^3$ (a generator of $\mathrm{H}^0(\mathcal{I}(1))$). Namely, the embedding $\mathrm{E} \longrightarrow \mathbb{P}(\mathrm{H}^0(\mathrm{E}, \mathcal{O}(\mathcal{D}))^*) = \mathbb{P}^2$ is projectively normal in view of Sekiguchi's results [9] and the above cohomology sequence becomes exact .

Analogously,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(2) & \longrightarrow & \mathcal{O}_{\mathrm{E}}(2) & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathcal{O}_{\mathrm{E}}(2\mathcal{D}) & & \end{array}$$

yields the exact sequence

$$(5)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathcal{I}(2)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(2)) & \longrightarrow & 0 \\ & & \dim = 4 & & \dim = 10 & & \dim = 6 & & \end{array}$$

also by projective normality. This indicates there are four quadratic equations; namely, the space

$$\mathrm{H} \otimes (\mathrm{U} + \mathrm{V} + \mathrm{W} - \alpha \mathrm{Z}).$$

To compute the dimension of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2))$ we have used that it is the space of degree 2 homogeneous polynomials in the variables $\{\mathrm{Z}, \mathrm{U}, \mathrm{V}, \mathrm{W}\}$, i.e. $\mathrm{S}^2(\mathrm{H})$.

Considering

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(3) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(3) & \longrightarrow & \mathcal{O}_{\mathrm{E}}(3) & \longrightarrow & 0\,, \\ & & & & & & \| & & \\ & & & & & & \mathcal{O}_{\mathrm{E}}(3\mathcal{D}) & & \end{array}$$

we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(\mathcal{I}(3)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(3)) & \longrightarrow & 0\,. \\ & & \dim = 11 & & \dim = 20 & & \dim = 9 & & \end{array}$$

This shows that there are 11 cubic equations for E. Ten of them are given by $\mathrm{S}^2(\mathrm{H}) \otimes (\mathrm{U} + \mathrm{V} + \mathrm{W} - \alpha \mathrm{Z})$, and there is a new one.

Now, since the degree of $\mathcal{O}_{\mathrm{E}}(\mathcal{D})$ is 3, we will show that E is the complete intersection of the linear and cubic equations. Indeed, $\dim \mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(n\mathcal{D})) = 3n$, $\dim \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(n)) =$ dim{homogeneous polynomials of degree n in $\mathbb{P}^3$} $= \binom{n+3}{3}$. On the other hand, the number of relations of order n generated by the linear and the cubic is $\binom{n+2}{3} + \binom{n}{3} - \binom{n-1}{3}$, which precisely coincides with $\binom{n+3}{3} - 3n$, and shows our statement.

3. Holomorphic vector fields on the elliptic curves

Let us write $u = \frac{\mathrm{U}}{\mathrm{Z}}, v = \frac{\mathrm{V}}{\mathrm{Z}}, w = \frac{\mathrm{W}}{\mathrm{Z}}$, for the affine coordinates.

The action of S_3 on these affine coordinates is given by the table

(6)

	u	v	w
σ	v	w	u
ι	u	w	v

Since the group S_3 is a subgroup of the Theta group associated with the polarization of $\mathcal{D}$, we have the invariance property of wronskians of sections of the line bundle $\mathcal{O}_{\mathrm{E}}(\mathcal{D})$ derivated with respect to the holomorphic vector field X on E. Namely $W_X(\sigma f, \sigma g) = \sigma W_X(f, g) = \sigma(f.X(g) - g.X(f))$, $W_X(\tau f, \tau g) = -\tau W_X(f, g)$ [6] (together with the relation $\dot{\mathrm{u}} = (\frac{\dot{\mathrm{U}}}{\mathrm{Z}}) = \frac{\mathrm{W_X(U,Z)}}{\mathrm{Z}^2}$ and similar ones for the other variables). We conclude that the vector field is quadratic in the affine variables. This follows because $W_X(f, g)$ belongs to $\mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(2\mathcal{D}))$ for $f, g \in \mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(\mathcal{D}))$, and as we have seen in (5), an element in $\mathrm{H}^0(\mathcal{O}_{\mathrm{E}}(2\mathcal{D}))$ comes from quadratic polynomials. Thus, we get the expression for the G-invariant generic vector field on E as follows:

$$\left\{\begin{array}{rcl} \dot{u} &=& \beta u(v-w) + \delta(v^2 - w^2) + \lambda(v-w) \\ \dot{v} &=& \beta v(w-u) + \delta(w^2 - u^2) + \lambda(w-u) \\ \dot{w} &=& \beta w(u-v) + \delta(u^2 - v^2) + \lambda(u-v) \end{array}\right. \tag{7}$$

The affine invariants up to third degree are
$u^3.+ v^3 + w^3$; $u^2(v+w) + v^2(w+u) + w^2(u+v)$; $u\,v\,w$;
$u^2 + v^2 + w^2$; $u\,v + v\,w + w\,u$; $u + v + w$.

It is easy to check that the ring of invariants $\mathbb{C}[u, v, w]^{S_3}$ is generated by $u+v+w$, $u\,v + v\,w + w\,u$, and $u\,v\,w$. So a cubic equation independent of the linear one, must have the expression:

$$\varphi = u\,v\,w + b\,(u\,v + v\,w + w\,u) = c. \tag{8}$$

This we deduce by writing a general cubic invariant under S_3 and using the obvious relations among the invariants. (Notice that since (7) is tangent to φ we get $\delta = 0$).

In this way we get a family of elliptic curves in $\mathbb{P}^3$ parametrized by parameters (a, b, c). Two such curves will be equivalent if there is an isomorphism preserving the divisors at infinity. This implies that there must be a linear map sending the affine piece of one curve into the affine piece of the other [6]. Moreover, if we require that the action of the group be preserved, we are calling for a nonsingular linear map $f : \mathbb{C}[u, v, w] \to \mathbb{C}[X, Y, Z]$ as follows:

(9)
$$\begin{cases} X &= a_1\,u + a_2\,v + a_2\,w \\ Y &= a_2\,u + a_1\,v + a_2\,w \\ Z &= a_2\,u + a_2\,v + a_1\,w \end{cases}$$

This linear map has to transform the curve defined by the ideal $\mathcal{I}_{(a,b,c)} =< \varphi_1(u,v,w) = u+v+w-a\,,\ \varphi_3(u,v,w) = u\,v\,w + b\,(u\,v + v\,w + w\,u) - c >$, into the curve defined by the ideal $\mathcal{I}_{(a',b',c')} =< \bar\varphi_1(X,Y,Z) = X+Y+Z-a'\,,\ \bar\varphi_3(X,Y,Z) = X\,Y\,Z + b'\,(X\,Y + Y\,Z + Z\,X) - c' >$.

Let us write $\varphi_{11}(u,v,w) = u+v+w$, $\varphi_{22} = u\,v + v\,w + w\,u$, $\varphi_{33}(u,v,w) = u\,v\,w$; (respectively $\bar\varphi_{ii}$ if they depend on the variables X,Y,Z).

We have:

(10)
$$\begin{aligned} f^*(\bar\varphi_{11}) &= (a_1 + 2\,a_2)\,\varphi_{11} \\ f^*(\bar\varphi_{22}) &= (a_1 - a_2)^2\,\varphi_{22} + a_2\,(a_2 + 2\,a_2)\,\varphi_{11}^2 \\ f^*(\bar\varphi_{33}) &= (a_1 - a_2)^3\,\varphi_{33} + a_2\,(a_1 - a_2)^2\,\varphi_{22}\,\varphi_{11} + a_2^2\,a_1\,\varphi_{11}^3\,, \end{aligned}$$

therefore, $f^*(\bar\varphi_1) = (a_1 + 2\,a_2)\,\varphi_{11} - a' = (a_1 + 2\,a_2)\,\varphi_1$ implies $a' = (a_1 + 2\,a_2)\,a$.

Also $f^*(\bar\varphi_3) = (a_1 - a_2)^3\,\varphi_{33} + a_2\,(a_1 - a_2)^2\,\varphi_{22}\,\varphi_{11} + a_2^2\,a_1\,\varphi_{11}^3 + b'\,((a_1 - a_2)^2\,\varphi_{22} + a_2\,(a_2 + 2\,a_1)\,\varphi_{11}^2) - c' = (a_1 - a_2)^3\,\varphi_3 - b\,(a_1 - a_2)^3\,\varphi_{22} + c\,(a_1 - a_2)^3 + a_2\,(a_1 - a_2)^2\,(\varphi_1 + a)\,\varphi_{22} + a_2^2\,a_1\,(\varphi_1 + a)^3 + b'\,((a_1 - a_2)^2\,\varphi_{22} + a_2\,(a_2 + 2\,a_1)\,(\varphi_1 + a)^2) - c'$ = Something in the ideal $\mathcal{I}_{(a,b,c)} + (-b\,(a_1 - a_2)^3 + a\,a_2\,(a_1 - a_2)^2 + b'\,(a_1 - a_2)^2)\,\varphi_{22} + (c\,(a_1 - a_2)^3 + a_2^2\,a_1\,a^3 + b'\,a_2\,(a_2 + 2\,a_1)\,a^2 - c')$; yields the relations among the parameters of the transformed curve and the original one:

(11)
$$\begin{aligned} a' &= (a_1 + 2\,a_2)\,a \\ b' &= (a_1 - a_2)\,b - a_2\,a \\ c' &= (a_1 - a_2)^3\,c + a_2\,(a_1 - a_2)\,(a_2 + 2\,a_1)\,b\,a^2 - a_2^2\,(a_1 + a_2)\,a^3\,. \end{aligned}$$

By eliminating a_1 and a_2 from (11) we get the following surface in the variables (a', b', c') whose points correspond to curves equivalent to the curve of parameters (a,b,c):

$$(a\,b^2+b^3+c)\,(a'^3+9\,a'^2\,b')-(a^3+9\,a^2\,b-27\,c)\,(a'\,b'^2+b'^3)-(a^3+9\,a^2\,b+27\,a\,b^2+27\,b^3)\,c' = 0.$$

4. The Poisson structures

We will look for a Poisson structure of the form $\{f,g\} =< \nabla f, J.\nabla g >$, where $J=(J_{ij})$ is a skew symmetric matrix with polynomial entries that must satisfy the Jacobi identity, amounting to:

$$\sum_k \frac{\partial J_{ij}}{\partial y_k} J_{kl} + \frac{\partial J_{jl}}{\partial y_k} J_{ki} + \frac{\partial J_{li}}{\partial y_k} J_{kj} = 0 \quad \forall i,j,l \tag{12}$$

where the y_k's take place of the variables u, v, w .

We will assume that the matrix

$$J = \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} \tag{13}$$

has polynomial entries and is invariant under the action of G. That is: $J(\sigma U) = \Lambda(\sigma)\, J(U)\, \Lambda(\sigma)^t$, $J(\iota U) = -\Lambda(\iota)\, J(U)\, \Lambda(\iota)^t$, where $U = (u, v, w)$, and $\Lambda(\sigma)$, $\Lambda(\iota)$ are the 3×3 matrices of σ and ι.

This implies $\alpha = \sigma\,\beta = \sigma^2\,\gamma$, $\beta = \iota\,\alpha$, $\gamma = \iota\,\gamma$, and if $\alpha \in \Sigma_{n=0}^N S^n(V) = L^N$, $V =$ linear span of (u, v, w), then $\beta,\ \gamma \in L^N$.

The algebra of hamiltonian functions will be $\mathbb{C}[\varphi_1, \varphi_2]$. So, after factoring out by the equations $\varphi_1 = 0$, $\varphi_3 = 0$, any expression $\{y_k, H\}$ (H a hamiltonian function) will be a linear combination of $\{y_k, \varphi_1\}$ and $\{y_k, \varphi_3\}$ with coefficients in $\mathbb{C}[a, b, c]$. However, we have the conditions that all hamiltonian vector fields must be quadratic in the variables (u, v, w). Namely $\{u, \varphi_1\} = \alpha - \beta \in L^2$, and $\{u, \varphi_3\} = \alpha\,(u\,w + b\,(u+w)) - \beta\,(u\,v + b\,(u+v)) \in L^2$. Moreover, we have the commutativity condition:

$$\{\varphi_3, \varphi_1\} = (1 + \sigma + \sigma^2) \cdot (\alpha\,(w + b)\,(v - u)) = 0\,, \tag{14}$$

and condition (12), that can be written

$$(1 + \sigma + \sigma^2) \cdot (\alpha \cdot (\sigma^2(\frac{\partial \alpha}{\partial v}) - \sigma(\frac{\partial \alpha}{\partial u})) = 0\,. \tag{15}$$

Lemma 2. *The only G-invariant Poisson structures are quadratic and have the form (13) with*

$$\alpha(u, v, w) = r\,(u\,v + b\,(u + v)) + s,$$

r and s parameters, $\alpha = \sigma\,\beta = \sigma^2\,\gamma$, $\beta = \iota\,\alpha$, $\gamma = \iota\,\gamma$.

Proof: Assume $\beta = c_1\,u^{\nu_1}\,v^{\nu_2}\,w^{\nu_3}$ + lower ordering terms with $\nu_1+\nu_2+\nu_3 \geq 3$. Then, in order to have $\alpha - \beta$ with quadratic terms, β (and α) has to contain the expression $c_1\,(\Sigma_{\sigma\in S_3} u^{\sigma\nu_1}\,v^{\sigma\nu_2}\,w^{\sigma\nu_3})$. Let us pick $c_1\,u^{\nu_1}\,v^{\nu_2}\,w^{\nu_3}$ the highest order term of β and α according to lexicographic monomial ordering. In the expression

$$\alpha\,(u\,w + b\,(u+w)) - \beta\,(u\,v + b\,(u+v)), \tag{16}$$

this term yields $c_1\,u^{\nu_1+1}\,v^{\nu_2}\,w^{\nu_3+1} - c_1\,u^{\nu_1+1}\,v^{\nu_2+1}\,w^{\nu_3}$. The monomial $u^{\nu_1+1}\,v^{\nu_2+1}\,w^{\nu_3}$ has the highest order and cannot be cancelled by a term coming from α. Thus, $c_1 = 0$ and we can eliminate all terms that are not quadratic. Therefore,

(17)

$$\begin{aligned}\beta &= a_{11}\,u^2 + a_{12}\,u\,v + a_{13}\,u\,w + a_{22}\,v^2 + a_{23}\,v\,w + a_{33}\,w^2 + b_1\,u + b_2\,v + b_3\,w + c_4\\ \alpha &= a_{11}\,u^2 + a_{12}\,u\,w + a_{13}\,u\,v + a_{22}\,w^2 + a_{23}\,v\,w + a_{33}\,v^2 + b_1\,u + b_2\,w + b_3\,v + c_4\,.\end{aligned}$$

Since the degree four polynomial in (16) has to vanish and degree four piece of (16) $= a_{11}\,u^3\,w + a_{12}\,u^2\,w^2 + a_{22}\,u\,w^3 + a_{23}\,u\,v\,w^2 + a_{33}\,u\,v^2\,w - a_{11}\,u^3\,v - a_{12}\,u^2\,v^2 - a_{22}\,u\,v^3 - a_{23}\,u\,w\,v^2 - a_{33}\,u\,v\,w^2$, we obtain $a_{23} = a_{33}$, $a_{11} = a_{12} = a_{22} = 0$. Thus:

$$\begin{aligned}\beta &= a_{13}\,u\,w + a_{23}\,v\,w + a_{23}\,w^2 + b_1\,u + b_2\,v + b_3\,w + c_4\\ \alpha &= a_{13}\,u\,v + a_{23}\,v\,w + a_{23}\,v^2 + b_1\,u + b_2\,w + b_3\,v + c_4\,.\end{aligned} \tag{18}$$

Now, we obtain the degree three piece of (16) $= (b_1 - b\,a_{13})\,u^2\,w + (b_2 - b\,a_{23})\,u\,w^2 + (b\,a_{13} - b_1)\,u^2\,v + (b\,a_{23} - b_2)\,v^2\,u$, yielding $b_1 = b\,a_{13}$, $b_2 = b\,a_{23}$, $\beta = a_{13}\,u\,(w+b) + a_{23}\,((v+w)\,w + b\,v) + b_3\,w + c_4$.

Using that $\alpha = \iota\,\beta = \sigma\,\beta$ and comparing the last two polynomials gives the final expression

$$\alpha = a_{13}\,(u\,v + b\,(u+v)) + c_4.$$

The relations (14) and (15) are immediately satisfied. □

Lemma 3. *The algebra of Casimir functions is generated by φ_1 if $a_{13} = 0$ and by $\varphi_3 + \frac{c_4}{a_{13}}\varphi_1$ if $a_{13} \neq 0$. In the first case a nontrivial hamiltonian is φ_3 and in the second case a nontrivial hamiltonian is φ_1.*

Proof: Let $p(\varphi_1, \varphi_3)$ be a Casimir where p is a polynomial of two variables. Then, p has to be a solution of the equation

$$(\alpha-\beta)\,\frac{\partial p}{\partial\varphi_1}+(\alpha\,\frac{\partial\varphi_3}{\partial v}-\beta\,\frac{\partial\varphi_3}{\partial w})\,\frac{\partial p}{\partial\varphi_3}=(u+b)\,(v-w)\,(a_{13}\,\frac{\partial p}{\partial\varphi_1}-c_4\,\frac{\partial p}{\partial\varphi_3})\,.$$

If $a_{13}=0$, $\frac{\partial p}{\partial\varphi_3}=0$ and p is a polynomial in φ_1.

If $a_{13}\neq 0$, then p satisfies $\frac{\partial p}{\partial\varphi_1}=\frac{c_4}{a_{13}}\frac{\partial p}{\partial\varphi_3}$. Let p_m be the m^{th} degree homogeneous polynomial of $p=\Sigma_{m=0}^{N}p_m$. Then, each p_m satisfies $\frac{\partial p_m}{\partial\varphi_1}=\frac{c_4}{a_{13}}\frac{\partial p_m}{\partial\varphi_3}$ and Euler's identity $\varphi_1\frac{\partial p_m}{\partial\varphi_1}+\varphi_3\frac{\partial p_m}{\partial\varphi_3}=mp_m$. Thus, $p_m=c_m\,(\varphi_3+\frac{c_4}{a_{13}}\varphi_1)^m$, which shows our statement. □

So far we get the following systems:

(19)

Casimirs generated by			$(u+v+w)$
Hamiltonian	φ_3	$=$	$u\,v\,w+b\,(u\,v+v\,w+w\,u)$
Vector Field	$\dot{u}$	$=$	$c_4\,u\,(v-w)+c_4\,b\,(v-w)$, cycle $\{u\to v\to w\to u\}$.

and

(20)

Casimirs generated by			$(u\,v\,w+b\,(u\,v+v\,w+w\,u))+\frac{c_4}{a_{13}}(u+v+w)$
Hamiltonian	φ_1	$=$	$(u+v+w)$
Vector Field	$\dot{u}$	$=$	$a_{13}\,u\,(v-w)+a_{13}\,b\,(v-w)$, cycle $\{u\to v\to w\to u\}$.

Now, we look for possible equivalences of the form (9) but preserving the algebra of Casimirs and the Poisson structure given by the matrix J. Namely $\{f^*(F),f^*(H)\}=f^*\{F,H\}$, $f(u,v,w)=(X,Y,Z)$. Let $\tilde{\alpha}(X,Y,Z)=\tilde{r}\,(X\,Y+\tilde{b}\,(X+Y))+\tilde{s}$, and $\alpha(u,v,w)=r\,(u\,v+b\,(u+v))+s$. At the level of the J matrices, preservation of the Poisson structure means

$$\begin{pmatrix}0&\tilde{\alpha}&-\tilde{\beta}\\-\tilde{\alpha}&0&\tilde{\gamma}\\\tilde{\beta}&-\tilde{\gamma}&0\end{pmatrix}=\begin{pmatrix}a_1&a_2&a_2\\a_2&a_1&a_2\\a_2&a_2&a_1\end{pmatrix}\cdot\begin{pmatrix}0&\alpha&-\beta\\-\alpha&0&\gamma\\\beta&-\gamma&0\end{pmatrix}\cdot\begin{pmatrix}a_1&a_2&a_2\\a_2&a_1&a_2\\a_2&a_2&a_1\end{pmatrix}=$$

$$(a_1-a_2).\begin{pmatrix}0&(a_1+a_2)\,\alpha-a_2\,(\beta+\gamma)&-((a_1+a_2)\,\beta-a_2\,(\alpha+\gamma))\\-((a_1+a_2)\,\alpha-a_2\,(\beta+\gamma))&0&(a_1+a_2)\,\gamma-a_2\,(\alpha+\beta)\\(a_1+a_2)\,\beta-a_2\,(\alpha+\gamma)&-((a_1+a_2)\,\gamma-a_2\,(\alpha+\beta))&0\end{pmatrix}$$

Thus, $\tilde{\alpha}(u,v,w) = \tilde{r}\,(a_1^2\,u\,v + a_1\,a_2\,(v^2+v\,w+u^2+u\,w) + a_2^2\,(v+w)\,(u+w) + \tilde{b}\,(a_1\,(u+v) + a_2\,(u+w) + a_2\,(v+w))) + \tilde{s}$ has to equal $(a_1-a_2)\,((a_1+a_2)\,(r\,(u\,v+b\,(u+v))+s) - a_2\,(r\,(u\,w+b\,(u+w))+s+r\,(v\,w+b\,(v+w))+s))$.

If $r=0$, then $\tilde{r}=0$ and $\tilde{s}=(a_1-a_2)^2\,s$. But $\dot{X} = a_1\,\dot{u} + a_2\,(\dot{v}+\dot{w})$ implies $\tilde{s}\,(X+\tilde{b})\,(Y-Z) = \tilde{s}\,(a_1\,u + a_2\,(v+w) + \tilde{b})\,(a_1-a_2)\,(v-w) = (a_1-a_2)^3\,s\,(a_1\,u + a_2\,(v+w) + (a_1-a_2)\,b - a_2\,a)\,(v-w) = (a_1-a_2)\,s\,(u+b)\,(v-w)$. That is $(a_1-a_2)^3 = 1$. Namely, equivalent systems depend on one parameter. After factoring out the parameters space (a,b,c,s) we obtain three effective parameters that characterize these systems.

If $r \neq 0$, also $\tilde{r} \neq 0$. Then $a_2 = 0$ and $\tilde{r}\,a_1\,(a_1\,u\,v + \tilde{b}\,(u+v)) + \tilde{s} = a_1^2\,(r\,(u\,v + b\,(u+v)) + s)$ implies $\tilde{r}=r$, $\tilde{s}=a_1^2 s$.

The vector field is written as $\dot{X} = \tilde{r}\,(X+\tilde{b})\,(Y-Z) = \tilde{r}\,a_1^2\,(u+b)\,(v-w)$ and $\dot{X} = a_1\,\dot{u} = a_1\,r\,(u+b)\,(v-w)$. Therefore $a_1 = 1$ and the parameters (a,b,c,r,s) are effective and describe all nonequivalent systems with quadratic Poisson matrix and linear hamiltonian.

Any polynomial Poisson bracket in $\mathbb{C}^3$ can be written as the sum of a quadratic Poisson bracket as above and a Poisson bracket with values in the ideal $\mathcal{I}_{(a,b,c)}$ (At least in terms of Hochschild cohomology of $\mathbb{C}[u,v,w]$). However a bracket with values in $\mathcal{I}_{(a,b,c)}$ is meaningless in terms of dynamics because it yields only trivial vector fields on the affine ring $\mathbb{C}[u,v,w]/\mathcal{I}_{(a,b,c)}$ of the elliptic curve.

The parameters (r,b,s) parametrize the different Poisson structures we put in $\mathbb{C}^3$. Here, for two Poisson manifolds to be equivalent we do not require the vector fields to map into one another. The J-matrices are preserved though. This leads to a two parameter space of nonequivalent Poisson manifolds for we take $s=1$.

References

[1] M. Adler, P. van Moerbeke *The complex geometry of the Kowalewski-Painlevé analysis.* Invent. math. **97** (1989), 3–51.

[2] Lange, H. - Birkenhake, Ch. *Complex Abelian Varieties.* Springer-Verlag - Grundlehren der Mathematischen Wissenschaften 302.

[3] Hartshorne, R. *Algebraic Geometry* Springer-Verlag - Graduate Texts in Mathematics 52.

[4] D. Mumford, *On the equations defining abelian varieties. I.* Inv. Math. **1** (1966), 287–354.

[5] Mumford, D. *Tata Lectures on Theta I.* Progress in Mathematics, Vol 28, Birkhäuser.

[6] Piovan, L. *Algebraically completely integrable systems and Kummer varieties.* Math. Ann. **290** (1991), no. 2, 349–403.

[7] Piovan, L. *Constructing a completely integrable system via algebro-geometric data.* J. Physics A (special issue) **34** (2001), no. 11, 2165–2178 .

[8] L. Piovan, P. Vanhaecke *Integrable systems and projective images of Kummer surfaces.* Annali della Scuala Normale Superiore de Pisa, **XXIX** (2000) 351–392. Preprint no. 131, Département de Mathématiques, Université de Poitiers.

[9] Sekiguchi, T. On the cubics defining abelian varieties J. Math. Soc. Japan **30** (1978), no. 4, 703–721.

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