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INJECTIVES IN QUASIVARIETIES OF POCRIMS

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ABSTRACT. Injectives and absolute subretracts in several classes of partially ordered commutative residuated integral monoids (pocrims) are characterized. Among the classes considered are residuated lattices, Girard monoids, bounded hoops, BL-algebras and Heyting algebras.

INTRODUCTION

Residuated structures, rooted in the work of Dedekind on the ideal theory of rings, arise in many fields of mathematics, and are particularly common among algebras associated with logical systems. They are structures $\langle A, \odot, \rightarrow, \leq \rangle$ such that A is a non empty set, $\leq$ is a partial order on A and $\odot$ and $\rightarrow$ are binary operations such that the following relation holds for each a, b, c in A:

$$a \odot b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

Important examples of residuated structures related to logic are Boolean algebras (corresponding to classical logic), Heyting algebras (corresponding to intuitionism), residuated lattices (corresponding to logics without contraction rule [13]), BL-algebras (corresponding to Hájek's basic fuzzy logic [10]), MV-algebras (corresponding to Łukasiewicz many-valued logic [5]). A common substructure of all these examples is the class of *partially ordered commutative residuated integral monoids*, or *pocrims* for short [4]. The aim of this paper is to investigate injectives and absolute subretracts in classes of pocrims.

After recalling in §1 some basic definitions and properties, we show in §2 that classes of pocrims satisfying some mild condition have only trivial absolute subretracts. In §3 we turn our attention to *bounded pocrims*, i. e., pocrims with a bottom element considered as zero-ary operation. We show that injectives in the categories of bounded pocrims and residuated lattices are trivial. Moreover, it is also shown that absolute subretracts are trivial in the categories of residuated lattices and Girard monoids. In §4 we consider injectives in subquasivarieties of bounded pocrims and in §5 we characterize the injectives in varieties of bounded hoops.

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1. Basic Notions

For all objects A, B in a category $\mathcal{A}$, $[A, B]_{\mathcal{A}}$ will denote the set of all morphism $g : A \to B$. Recall that an object A of a category $\mathcal{A}$ is **injective** iff for every monomorphism $f \in [B, C]_{\mathcal{A}}$ and every $g \in [B, A]_{\mathcal{A}}$ there exists $h \in [C, A]_{\mathcal{A}}$ such that $hf = g$. An object B is a **retract** of an object A iff there exist $g \in [B, A]_{\mathcal{A}}$ and $f \in [A, B]_{\mathcal{A}}$ such that g is a monomorphism, f is an epimorphism, and $fg = 1_B$. An object B is called **absolute subretract** in $\mathcal{A}$ iff it is a retract of each of its extensions in $\mathcal{A}$. It is well known that, a retract of an injective object is injective, and injectives are absolute subretracts. For each algebra A we denote by $Con(A)$, the congruence lattice of A. An algebra I is **simple** iff $Con(I) = \{\Delta, \nabla\}$.

A **pocrim** [4] is an algebra $\langle A, \odot, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ satisfying the following axioms:

1. $\langle A, \odot, 1\rangle$ is an abelian monoid,
2. $x \to 1 = 1$,
3. $1 \to x = x$,
4. $(x \to y) \to ((z \to x) \to (z \to y)) = 1$,
5. $x \to (y \to z) = (x \odot y) \to z$,
6. If $\quad x \to y = 1 \quad$ and $\quad y \to x = 1 \quad$ then $\quad x = y$.

We denote by $\mathcal{M}$ the class of all pocrims. $\mathcal{M}$ is a quasivariety and does not conform a variety [9]. For all $A \in \mathcal{M}$, the relation $\leq$ on A defined by $x \leq y$ iff $x \to y = 1$, makes $\langle A, \odot, \to, 1, \leq\rangle$ into a conmutative partial ordered monoid in which 1 is the upper bound. An element $x \in A$ is called **idempotent** iff $x \odot x = x$, and the set of all idempotent elements in A is denoted by $Idp(A)$. We also define for all $a \in A$, $a^1 = a$ and $a^{n+1} = a^n \odot a$. It is easy to verify the following proposition:

Proposition 1.1. *The following assertions hold in every pocrim A, where x, y, z denote arbitrary elements of A:*

(1) $x \to x = 1$.
(2) $x \to (y \to z) = y \to (x \to z)$.
(3) *If* $x \leq y$ *then* $\quad x \to z \geq y \to z \quad$ *and* $\quad z \to y \leq z \to x$.
(4) $x \odot y \leq z$ *iff* $x \leq y \to z$.
(5) $x \odot y \leq y$.
(6) $x \odot (x \to y) \leq y$.

□

We recall now some well known facts about filters and congruences. Let A be a pocrim. A set $F \subseteq A$ is an **implicative filter** iff F satisfies the following conditions:

1. $1 \in F$,
2. If $x \in F$ and $x \to y \in F$ then $y \in F$.

It is easy to verify that $F \subseteq A$ is an implicative filter iff $1 \in F$ and for all $a, b \in A$:

- If $a \in F$ and $a \leq b$ then $b \in F$,
- if $a, b \in F$ then $a \odot b \in F$.

For every filter F of A, $\theta_F = \{(x, y) \in A^2 : x \to y \text{ and } y \to x \in F\}$ is a congruence in A and A/θ_F is a pocrim. We denote by $\langle X\rangle$ the implicative filter generated by $X \subseteq A$, i.e. the intersection of all implicative filter of A containing X. We abbreviate $\langle a \rangle$ when $X = \{a\}$ and it is easy to verify that $\langle a \rangle = \{x \in A : \exists n \geq 1 \textit{ such that } x \geq a^n\}$. The set $Filt(A)$ of all implicative filtes of A, ordered by inclusion, is a bounded lattice.

A **hoop** [4] is a pocrim satisfying $x \leq y$ iff $x = (x \to y) \odot x$. Every hoop is a meet semilattice, where the meet operation is given by $x \wedge y = x \odot (x \to y)$. Hoops form a variety $\mathcal{HO}$ defined by the following equations:

1. $\langle A, \odot, 1 \rangle$ is an abelian monoid,
2. $x \to x = 1$,
3. $(x \to y) \odot x = (y \to x) \odot y$,
4. $x \to (y \to z) = (x \odot y) \to z$.

Let k be a natural number. A **k-potent hoop** [4] is a hoop satisfying $x^k = x^{k+1}$. We denote the class of k-potent hoop by $\mathcal{HO}(k)$. It is clear that $\mathcal{HO}(2)$ is the variety of Brouwerian semilattices [14].

A **basic hoop** [1] is an algebra $\langle A, \wedge, \vee, \odot, \to, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0 \rangle$ such that:

1. $\langle A, \odot, \to, 1 \rangle$ is a hoop,
2. $\langle A, \wedge, \vee, 1 \rangle$ is lattice with greatest element 1,
3. $(x \to y) \vee (y \to x) = 1$.

Basic hoops are also known as **generalized BL-algebras** [6]. We denote by $\mathcal{BH}$ the category whose elements are basic hoops.

2. Absolute subretracts in pocrims

The proof of the next proposition is immediate and will be omitted.

Proposition 2.1. *Let A be a pocrim and $\perp$ be a new symbol not belonging to A. We can consider $\perp \oplus A = A \cup \{\perp\}$ with the following operation:*

$$x \odot_{\perp} y = \begin{cases} x \odot y, & \text{if } x, y \in A \\ \perp, & \text{if } x = \perp \text{ or } y = \perp \end{cases}$$

$$x \to_{\perp} y = \begin{cases} x \to y, & \text{if } x, y \in A \\ \perp, & \text{if } x \in A \text{ and } y = \perp \\ 1, & \text{if } x = \perp \end{cases}$$

Then $\langle \perp \oplus A, \odot_{\perp}, \to_{\perp}, 1 \rangle$ is a pocrim with smallest element $\perp$, and A is a subalgebra of $\perp \oplus A$. □

Definition 2.2. *Let $\mathcal{A}$ be a subquasivariety of $\mathcal{M}$. We say that $\mathcal{A}$ is* **$(\perp \oplus)$-closed** *iff for all $A \in \mathcal{A}$, $\perp \oplus A \in \mathcal{A}$*

Theorem 2.3. *If $\mathcal{A}$ is a $(\perp \oplus)$-closed subquasivariety of $\mathcal{M}$, then absolute subretracts in $\mathcal{A}$ are trivial algebras.*

Proof: Suppose that there exists a non trivial absolute subretract A in $\mathcal{A}$. Let $i : A \to \perp \oplus A$ be the monomorphism such that $i(x) = x$. Then there exists an epimorphism $f : \perp \oplus A \to A$ such that the composition $fi = 1_A$. Let $0 = f(\perp)$. Since for all $x \in A$, $0 = f(\perp) \leq f(i(x)) = x$, we have that 0 is the smallest element of A. In $\perp \oplus A$ we have that $0 \to \perp = \perp$. Therefore $f(0 \to \perp) = f(\perp) = 0$. On the other hand, since $i(0) = 0$, $f(0) \to f(\perp) = 0 \to 0 = 1$. Hence $0 = f(0 \to \perp)$ but $f(0) \to f(\perp) = 1$, which is a contradiction. Consequently $\mathcal{A}$ has only trivial absolute subretracts. □

Corollary 2.4. *If $\mathcal{A}$ is a $(\perp \oplus)$-closed subquasivariety of $\mathcal{M}$, then $\mathcal{A}$ has only trivial injectives.* □

Corollary 2.5. *$\mathcal{M}$, $\mathcal{HO}$, $\mathcal{HO}(k)$, $\mathcal{BH}$ have only trivial absolute subretracts and trivial injectives.* □

3. Bounded pocrims and residuated lattices

A **bounded pocrim** is an algebra $\langle A, \odot, \to, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that:

1. $\langle A, \odot, \to, 1\rangle$ is a pocrim
2. $0 \to x = 1$

In every bounded pocrim we can define a unary operation $\neg$ by $\neg x := x \to 0$. The quasivariety whose elements are bounded pocrims, is noted by $\mathcal{M}_0$. Observe that since 0 is in the clone of operation, then we require that for each morphism f, $f(0) = 0$. An element $a < 1$ in a bounded pocrim is called **nilpotent** iff there exist a natural number n such that $a^n = 0$.

Proposition 3.1. *Let A be a bounded pocrim and $\theta \in Con(A)$. Then, $\theta \neq \nabla$ iff $(0, 1) \notin \theta$.*

Proof: For the non trivial part, suppose that $(0, 1) \in \theta$. Let $(a, b) \in A^2$. Since $(a, a) \in \theta$ and $(b, b) \in \theta$ then $(1 \to a, 0 \to a) = (a, 1) \in \theta$ and $(0 \to b, 1 \to b) = (1, b) \in \theta$. Thus $(a \odot 1, 1 \odot b) = (a, b) \in \theta$ and then θ is not proper. □

An important subclass of $\mathcal{M}_0$ is the variety of **residuated lattices** [13]. A residuated lattice is an algebra $\langle A, \wedge, \vee, \odot, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ satisfying the following axioms:

1. $\langle A, \odot, 1\rangle$ is an abelian monoid,
2. $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice,
3. $(x \odot y) \to z = x \to (y \to z)$,
4. $((x \to y) \odot x) \wedge y = (x \to y) \odot x$,
5. $(x \wedge y) \to y = 1$.

The variety of residuated lattices is noted by $\mathcal{RL}$. A **Girard-monoid** [11] is a residuated lattice characterized by the equation $\neg\neg x = x$. The variety of Girard-monoids is noted by $\mathcal{GM}$. If A is a residuated lattice and $\theta \in Con(A)$, it is easy to verify that $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is an implicative filter. Moreover, the map $\theta \to F_\theta$ establishes an order anti-isomorphism between $Con(A)$ and $Filt(A)$. It is easy to verify the following propositions:

Proposition 3.2. *Let A be a residuated lattice, then the following assertions are equivalent*

1. *A is simple,*
2. *For $a \in A$, if $a < 1$ then a is nilpotent.* □

Proposition 3.3. *Let A be a totally ordered set with smallest element 0, greatest element 1 and coatom u. If we consider the following operations in A:*

$$x \odot y = \begin{cases} 0, & if\ x, y < 1 \\ x, & if\ y = 1 \\ y, & if\ x = 1 \end{cases}$$

$$x \to y = \begin{cases} 1, & if\ x \leq y \\ y, & if\ x = 1 \\ u, & if\ y < x < 1 \end{cases}$$

then we have

1. *$\langle A, \odot, \to, 0, 1\rangle$ is a simple bounded pocrim,*
2. *$\langle A, \wedge, \vee, \odot, \to, 0, 1\rangle$ is a simple residuated lattice.*

Proof: 1) Let θ be a non trivial congruence and $(a, b) \in \theta$ such that $a < b$. Since $(b, b) \in \theta$ we have that $(b \to a, b \to b) = (u, 1) \in \theta$ and then $(u \odot u, 1 \odot 1) = (0, 1) \in \theta$. By Proposition 3.1, θ is a proper congruence, which is a contradiction. Consequently A is simple in $\mathcal{M}_0$. 2) It is easy to verify that A satisfies the residuated lattice equations and by Proposition 3.2 A is simple in $\mathcal{RL}$. □

Definition 3.4. *An ordinal pocrim (residuated lattice) is an ordinal $\gamma = Suc(Suc(\alpha))$, for some ordinal α, with the structure given by Proposition 3.3. In this case α is the coatom in γ.*

Theorem 3.5. *Let $\mathcal{A}$ be a subquasivariety of $\mathcal{M}_0$ ($\mathcal{RL}$) such that $\mathcal{A}$ contain the ordinal pocrims (residuated lattices). Then $\mathcal{A}$ have only trivial injectives.*

Proof: Suppose that there exists a non trivial injective A in $\mathcal{A}$. Let α be a cardinal such that $\alpha > Card(A)$. We consider the ordinal pocrims (residuated lattices) $\gamma = Suc(Suc(\alpha))$ and $2 = Suc(Suc(\emptyset)) = 0, 1$. Let $i_A : 2 \to A$ and $i_A : 2 \to \gamma$ be a trivial embedding. Since A is injective and γ is a simple algebra, there exists a monomorphism $\varphi : \gamma \to A$ such that $\varphi i_\gamma = i_A$. Thus $\gamma \leq Card(A) < \alpha \leq \gamma$ which is a contradiction. Consequently $\mathcal{A}$ have only trivial injectives. □

Corollary 3.6. *$\mathcal{M}_0$ and $\mathcal{RL}$ have only trivial injectives.* □

For residuated lattices we can give a more general result in terms of the absolute subretracts.

Proposition 3.7. *Let A be a residuated lattice. Then the set $A^\diamond = \{(a, b) \in A \times A : a \leq b\}$ equipped with the operations:*

$$(a_1, b_1) \wedge (a_2, b_2) := (a_1 \wedge a_2, b_1 \wedge b_2),$$

$(a_1, b_1) \vee (a_2, b_2) := (a_1 \vee a_2, b_1 \vee b_2),$
$(a_1, b_1) \odot (a_2, b_2) := (a_1 \odot a_2, (a_1 \odot b_2) \vee (a_2 \odot b_1)),$
$(a_1, b_1) \rightarrow (a_2, b_2) := ((a_1 \rightarrow a_2) \wedge (b_1 \rightarrow b_2), a_1 \rightarrow b_2).$

is a residuatedd lattice, and the following properties hold:

1. $i : A \rightarrow A^\diamond$ *defined by* $i(a) = (a, a)$ *is a monomorphism.*
2. $\neg(a, b) = (\neg b, \neg a)$ *and* $\neg(0, 1) = (0, 1)$.
3. A *is a Girard-monoid iff* $A^\diamond$ *is a Girard-Monoid.*

Proof: See [11, IV Lemma 3.2.1] □

Definition 3.8. *We say that a subvariety $\mathcal{A}$ of $\mathcal{RL}$ is* **$\diamond$-closed** *iff for all $A \in \mathcal{A}$, $A^\diamond \in \mathcal{A}$.*

Theorem 3.9. *If a subvarietiy $\mathcal{A}$ of $\mathcal{RL}$ is* **$\diamond$-closed** *then $\mathcal{A}$ has only trivial absolute subretracts.*

Proof: Suppose that there exist a non trivial absolute subretract A in $\mathcal{A}$. Then by Proposition 3.7 there exists an epimorphism $f : A^\diamond \rightarrow A$ such that the following diagram is conmutative

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "i"'] \arrow[dr, phantom, "\equiv"] & A \\ A^\diamond \arrow[ur, "f"'] & \end{tikzcd}$$

Thus there exists $a \in A$ such that $f(0, 1) = a = f(a, a)$. Since $(0, 1)$ is a fixpoint of the negation in $A^\diamond$ it follows that $0 < a < 1$. We have that $f : (a, 1) = 1$. Indeed, $(0, 1) \rightarrow (a, a) = ((0 \rightarrow a) \wedge (1 \rightarrow a), 0 \rightarrow a) = (a, 1)$. Thus $f(a, 1) = f((0, 1) \rightarrow (a, a)) = f(0, 1) \rightarrow f(a, a) = a \rightarrow a = 1$. In view of this we have that $1 = f(a, 1) \odot f(a, 1) = f((a, 1) \odot (a, 1)) = f(a \odot a, (a \odot 1) \vee (a \odot 1)) = f((a \odot a, a)) \leq f((a, a)) = a$, which is a contradiction since $a < 1$. Hence $\mathcal{A}$ has only trivial absolute subretracts. □

Corollary 3.10. *$\mathcal{RL}$ and $\mathcal{GM}$ has only trivial absolute subretracts.* □

4. Injectives in quasivarieties of bounded pocrims

An element a of a bounded pocrim A is said to be **dense** iff $\neg a = 0$. The set of all dense element of A is denoted by $Ds(A)$. It is easy to verify that $Ds(A)$ is an implicative filter. We are going to use the following notation: for each $x \in A$, $[x]$ will denote the $Ds(A)$-conguence class of x. An algebra A in $\mathcal{A}$ is said **dense free** iff $Ds(A) = \{1\}$. We denote by $\mathcal{DF}(\mathcal{A})$ the subclass of $\mathcal{A}$ whose elements are the dense free algebras of $\mathcal{A}$. It is easy to verify the following proposition:

Proposition 4.1. *Let $\mathcal{A}$ be a subquasivariety of $\mathcal{M}_0$.*

1. $\mathcal{DF}(\mathcal{A}) = \{A/Ds(A) : A \in \mathcal{A}\}$
2. $\mathcal{DF}(\mathcal{A})$ *is the subquasivariety of $\mathcal{A}$ characterized by the quasiequation* $\neg\neg x = 1 \Rightarrow x = 1$.

□

Proposition 4.2. *Let $\mathcal{A}$ be a subquasivariety of $\mathcal{M}_0$. Then $\mathcal{DF}(\mathcal{A})$ is a reflective subcategory, and the reflector preserves monomorphisms.*

Proof: For all $A \in \mathcal{A}$, define $\mathcal{DF}(A) = A/Ds(A)$, and for each $f \in [A, B]_{\mathcal{A}}$, define $\mathcal{DF}(f)$ such that $\mathcal{DF}(f)([x]) = [f(x)]$ for each $x \in A$. Since homomorphisms preserve dense elements, we obtain a well defined function $\mathcal{DF}(f) : A/Ds(A) \to B/Ds(B)$. It is easy to check that $\mathcal{DF}$ is indeed a functor from $\mathcal{A}$ to $\mathcal{DF}(\mathcal{A})$. To show that $\mathcal{DF}$ is a reflector, note first that if $p_A : A \to A/Ds(A)$ is the natural morphism, then the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\equiv"] & A' \arrow[d, "p_{A'}"] \\ A/Ds(A) \arrow[r, "\mathcal{DF}(f)"'] & A/Ds(A') \end{tikzcd}$$

Suppose now that $B \in \mathcal{DF}(\mathcal{A})$ and $f \in [A, B]_{\mathcal{A}}$. Since $Ds(B) = \{1\}$, the mapping $[x] \mapsto f(x)$ defines a homomorphism $g : A/Ds(A) \to B$ that makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_A"'] & B \\ A/Ds(A) \arrow[ur, "g"'] \arrow[u, phantom, shift left=4ex, "\equiv"] & \end{tikzcd}$$

and it is obvious that g is the only morphism in $[A/Ds(A), B]_{\mathcal{DF}(\mathcal{A})}$ making the triangle commutative. Therefore we have proved that $\mathcal{DF}$ is a reflector. We proceed to prove that $\mathcal{DF}$ preserves monomorphisms: let $f \in [A, B]_{\mathcal{A}}$ be a monomorphism and suppose that $(\mathcal{DF}(f))(x) = (\mathcal{DF}(f))(y)$, i.e., $[f(x)] = [f(y)]$. Then $0 = \neg(f(x) \to f(y)) = f(\neg(x \to y))$. Since f is a monomorphism, $\neg(x \to y) = 0$ and $x \to y \in Ds(A)$. Interchanges x and y, we obtain $[x] = [y]$ and $\mathcal{DF}(f)$ is monomorphism. □

Proposition 4.3. *Let $\mathcal{A}$ be a $(\bot\, \oplus)$-closed subquasivariety of $\mathcal{M}_0$. If B is injective in $\mathcal{A}$ then $Ds(B) \cap Idp(B) = \{1\}$.*

Proof: Let B be an injective in $\mathcal{A}$. If there is an element $a \in Ds(B) \cap Idp(B)$ with $a < 1$, then $\{0, a, 1\}$ would be a subalgebra of B such that $Ds(B) = B \backslash \{0\}$. Extend it to a maximal totally ordered subalgebra C of B such that $Ds(C) = C \backslash \{0\}$, and let $i_C : C \to B$ be defined by $i_C(x) = x$. In the algebra $\bot \oplus C$ we have $\bot < 0$. To avoid confusion, we define $\alpha := 0$. Now we define $f : C \to \bot \oplus C$ such that $f(0) = \bot$ and for each $x > 0$, $f(x) = x$. It is easy to verify that f is a monomorphism. Since B is injective there exists a morphism $g : \bot \oplus C \to B$ such that $gf = i_C$ since B is an injective object. We derive from this the following assertions:

1. $g(\alpha) \in C$ (since C is a maximal subchain of B with the property $Ds(C) = C - \{0\}$),
2. $g(\alpha) \neq 0$ (since $\neg g(\alpha) = g(\neg\alpha) = g(\bot) = 0$),

3. $g(\alpha) < 1$ (since $\alpha < a$ and then $g(\alpha) \leq g(a) = a < 1$).

Now we have that for all $x \in C - \{0\}$, $x \to g(\alpha) = g(x) \to g(\alpha) = g(x \to \alpha) = g(\alpha) < 1$ by item (3). Thus $g(\alpha) < x$. Hence by item (1) we obtain $g(\alpha) < g(\alpha)$ which is an obvious contradiction. Therefore we conclude that $Ds(B) \cap Idp(B) = \{1\}$. □

Proposition 4.4. *Let $\mathcal{A}$ be a $(\perp \oplus)$-closed subquasivariety of $\mathcal{M}_0$. If B is injective in $\mathcal{A}$ then $Ds(B) = \{1\}$.*

Proof: Let B be an injective in $\mathcal{A}$. We assume that there is an element $a \in Ds(B)$ with $a < 1$. For all natural number $n \geq 1$, $\neg(a^n) = 0$ since $\neg(a^n) = a^n \to 0 = a^{n-1} \to (a \to 0) = a^{n-1} \to 0 = \cdots = a \to 0 = 0$. Thus $a^n > 0$ for all $n \geq 1$, and then the principal implicative filter $\langle a \rangle$ is proper. Let $A = \langle a \rangle \cup \{0\}$. A is closed by $\neg$ since if $x = 0$ then $\neg x = 1$ and for $x \in \langle a \rangle$ there is exist $n \geq 1$ sucht that $x \geq a^n$ and then $\neg x \leq \neg(a^n) = 0$. Since $\langle a \rangle$ is an implicative filter, this proves that $A \in \mathcal{M}_0$. Let $A_\perp = \perp \oplus A$ and let $g : A \to A_\perp$ be the monomorphism such that $g(0) = \perp$ and $g(x) = x$ if $x \in \langle a \rangle$. Since B is injective, there is exist a morphism $f : A_\perp \to B$ such that:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv"] & B \\ A_\perp \arrow[ur, "f"'] & \end{tikzcd}$$

$f(0) \in Ds(B)$ since $\neg f(0) = f(\neg 0) = f(0 \to \perp) = f(\perp) = 0$, and $f(0) < 1$ since $f(0) \leq f(a) = 1_A(a) = a < 1$. Moreover $f(0) \in Idp(B)$ since $f(0) \odot f(0) = f(0 \odot 0) = f(0)$. Thus $f(0) \in Ds(B) \cap Idp(B)$ which is a contadition by Proposition 4.3. Therefore $Ds(B) = \{1\}$. □

Theorem 4.5. *Let $\mathcal{A}$ be $(\perp \oplus)$-closed subquasivariety of $\mathcal{M}_0$. Then A is injective in $\mathcal{A}$ iff A is injective in $\mathcal{DF}(\mathcal{A})$.*

Proof: If A is injective in $\mathcal{A}$ then by proposition 4.5 $Ds(A) = \{1\}$, thus $A \in \mathcal{DF}(\mathcal{A})$ and A is injective in $\mathcal{A}/Ds$. Conversaly by Propositions 4.2 since $\mathcal{DF}(\mathcal{A})$ is a reflective subcategory of $\mathcal{A}$ and the reflector preserves monomorphism. It is well known that if $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$ such that the reflector preserves monomorphisms then an injective object in $\mathcal{B}$ is also injective in $\mathcal{A}$ [2, I.18]. Thus A is injective in $\mathcal{DF}(\mathcal{A})$ then A is injective in $\mathcal{A}$. □

5. Injectives in varieties of bounded hoops

A **bounded hoop** is a bounded pocrim $\langle A, \odot, \to, 0, 1 \rangle$ such that $\langle A, \odot, \to, 1 \rangle$ is a hoop. It is clear that the class $\mathcal{HO}_0$ of bounded hoops is a variety contained in $\mathcal{M}_0$.

Lemma 5.1. *Let A be a bounded hoop, then the following assertions are valid:*

1. $x \odot \neg x = 0$,
2. $\neg(\neg\neg x \to x) = 0$ *i.e.* $\neg\neg x \to x \in Ds(A)$,
3. $x = \neg\neg x \odot (\neg\neg x \to x)$.

Proof: 1) $x \odot \neg x = x \odot (x \to 0) = x \wedge 0 = 0$. 2) It is the same argument used in [7, Lemma 1.3]. 3) $x \leq \neg\neg x$ since $x \odot \neg x = 0$, then $x = x \wedge \neg\neg x = x \odot (\neg\neg x \to x)$. □

A **BL-algebra** [10] (or **bounded basic hoop** [1]) is a residuated lattice which satisfies the following equations:

$$(i)\ x \wedge y = x \odot (x \to y) \quad \text{and} \quad (ii)\ (x \to y) \vee (y \to x) = 1.$$

The variety of BL-algebras is noted by $\mathcal{BL}$. A **pseudocomplemented BL-algebra** is a BL-algebra characterized by the equation $x \wedge \neg x = 0$. The variety of pseudocomplemented BL-algebra is noted by $\mathcal{PBL}$. A **MV-algebra** [5, 10] is a BL-algebra characterized by the equation $\neg\neg x = x$. The variety of MV-algebras is noted by $\mathcal{MV}$.This variety is generated by the MV-algebra $R_{[0,1]} = \langle [0,1], \odot, \to, \wedge, \vee, 0, 1 \rangle$ sucht that $[0,1]$ is the real unit segment, $\wedge$, $\vee$ are the natural meet and join on $[0,1]$ and $\odot$ and $\to$ are defined as follows: $x \odot y := max(0, x + y - 1)$, $x \to y := min(1, 1 - x + y)$. **Heyting algebras** [2] are residuated lattices characterized by the equation $x \odot y = x \wedge y$. The variety of Heyting-algebras is noted by $\mathcal{H}$. **linear Heyting algebras**, (also known as Gödel algebras [10]) are Heyting algebras satisfying the equation $(x \to y) \vee (y \to x) = 1$. The variety of Linear Heyting algebras is noted by $\mathcal{HL}$. We denote by $\mathcal{B}$ the variety of boolean algebras.

Lemma 5.2. *Let A be a residuated lattice, then the following assertions are equivalent*

1. *A is an MV-algebra.*
2. *A is Girard-monoid which satisfies the equations $x \wedge y = x \odot (x \to y)$.*

Proof: See [11, IV Lemma 2.14] and [12, VI Lemma 2.3] □

Proposition 5.3. *If $A \in \mathcal{HO}_0$ then $\mathcal{DF}(A)$ is a Girard-monoid.*

Proof: Let $A \in \mathcal{A}$ and $[x] \in A/Ds(A)$. By lemma 5.1 we have that $[x] = [\neg\neg x] \odot [\neg\neg x \to x]$ and $\neg\neg x \to x \in Ds(A)$, thus $[\neg\neg x \to x] = [1]$ then $[x] = [\neg\neg x]$ i.e. $A/Ds(A)$ is a Girard-monoid. □

Corollary 5.4. 1. $\mathcal{DF}(\mathcal{HO}_0) = \mathcal{DF}(\mathcal{BL}) = \mathcal{MV}$.
2. $\mathcal{DF}(\mathcal{PBL}) = \mathcal{DF}(\mathcal{H}) = \mathcal{DF}(\mathcal{HL}) = \mathcal{B}$.

Proof: $\mathcal{DF}(\mathcal{HO}_0)$ and $\mathcal{DF}(\mathcal{BL})$ is $\mathcal{MV}$ since their elemets are Girard-monoid satisfying the equation $x \wedge y = x \odot (x \to y)$ (Lemma 5.2). The other equalities are immediate. □

Proposition 5.5. 1. *A is injective in $\mathcal{HO}_0$ or $\mathcal{BL}$ iff A is a retract of a power of the MV-algebra $R_{[0,1]}$.*
2. *A is injective in $\mathcal{PBL}$, $\mathcal{H}$ or $\mathcal{HL}$ iff A is a complete boolean algebra.*

Proof: Since all these classes are $(\perp \oplus)$-closed, the results follow from Theorem 4.5, Corolary 5.4 and the well known characterization of injective MV-algebras (see [8, Corollary 2.11]) and injective boolean algebras [15]. □

Notice that the above caracterization of injective Heyting algebras was already given in [3].

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