ACTAS DEL VII CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2003) , Páginas 17–22

RESONANT PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

P. AMSTER, P. DE NÁPOLI AND M.C. MARIANI

ABSTRACT. We review some aspects of the theory of resonant ordinary differential equations, and present some recent results about higher order resonant differential equations.

1. INTRODUCTION

In the last years, there has been an increasing interest in resonant problems, both for ordinary and partial differential equations. In particular, the existence of periodic solutions for resonant ordinary equations has been widely studied. In this work, we review some aspects of the theory of second order resonant equations, and present some recent results about higher order resonant equations.
Many problems in nonlinear analysis can be written in the form,

$$Lx = Nx$$

where L is a linear differential operator, defined in a suitable functional space, and N is a Nonlinear operator (involving lower some order terms). The problem is called non resonant when the operator L is invertible. In that case the problem can be reduced to a fixed point problem,

$$x = L^{-1}Nx$$

and one can use Leray-Schauder degree theory, or fixed point theorems. When L is not invertible, the problem is called resonant. In that case, Mawhin coincidence degree theory can be used (see section 3).

For second order scalar equations, the following result is well known (see [21],[14]) :

Theorem 1.1. *Assume that $c > 0$, $p \in C(\mathbb{R})$ is 2π-periodic and that $g \in C(\mathbb{R})$ has limits at infinity*

$$g(\pm) = \lim_{x \to \pm} g(x) \tag{1}$$

Then the resonant second order equation

$$\ddot{x} + c\dot{x} + g(x) = p(t)$$

has a 2π-periodic solution if

$$g(-\infty) < \overline{p} = \frac{1}{2\pi} \int_0^{2\pi} p(t) \ < g(+\infty) \tag{2}$$

Moreover, if g satisfies

$$g(-\infty) < g(x) < g(+\infty) \; \forall \, x \in \mathbb{R} \tag{3}$$

then (2) is also necessary for the existence of 2π-periodic solutions.

The following closely related result was obtained by D.E. Leach and A. Lazer in [15]

Theorem 1.2. *Consider the second order ordinary differential equation*

$$\ddot{x} + m^2 x + g(x) = p(t)$$

where $m > 0$ is an integer and g satisfies the same conditions as before. If we consider the m-th Fourier coefficients

$$a_m(p) = \int_0^{2\pi} p(t) \; \cos(mt) \; dt$$

$$b_m(p) = \int_0^{2\pi} p(t) \; \sin(mt) \; dt$$

then the inequality

$$\sqrt{a_m(p)^2 + b_m(p)^2} < 2(g(+\infty) - g(-\infty)) \tag{4}$$

is both necessary and sufficient for the existence of 2π-periodic solutions.

Conditions like (2) and (4) are typical in many results on the existence of solutions for resonant problems. In the literature, they are known as Landesman-Lazer type conditions, after the pioneering work of these authors on the resonant Dirichlet problem for elliptic second order equations ([12]).

Finally, we remark that there are many interesting results for resonant systems of second order ordinary differential equations. In [21] some Landesman-Lazer conditions for systems are discussed. In [3], the case of a resonant second order system with a multidimensional kernel and periodic nonlinearities is studied.

2. Results for Higher Order Equations

Higher order equations are interesting both because of their applications (for example to multi-ion electrodiffussion problems [16] ; or in beam theory [8], [10]), and because of their intrinsic mathematical interest, since many tools that are usually applied to the study of second order equations are not available. For example, it is not possible, in general, to use the method of upper and lower solutions, since it depends on the maximum principle (see however [9]).

In [1], the authors have proved the following result for the case of a third order equation:

Theorem 2.1. *Consider the equation*

$$x''' + ax'' + \tilde{g}(x') + cx = p(t) \tag{5}$$

when $\tilde{g}(s) = \lambda s + g(s)$ *for bounded* g, $\lambda = m^2$ $(m \in \mathbb{Z})$ *and* $c = am^2 \neq 0$. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous bounded function such that the limits (1) exist and let* $p \in L^2(0, 2\pi)$. *If furthermore, we assume that*

$$a_m^2(p) + b_m^2(p) < \frac{4}{\pi^2}[g(+\infty) - g(-\infty)]^2 \tag{6}$$

then equation (5) has at least one 2π*-periodic solution in* $H^3(0, 2\pi)$.

Some previous results for third-order equations appeared in [4] and [5]. In [11], the case of a non-resonant fourth-order equation is studied.

This result was generalized in [2] for higher order differential equations. There, we have considered the problem:

$$Lx + g(x, x', \ldots, x^{(N-2)}) = p(t) \tag{7}$$

where

$$Lx = x^{(N)} + a_{N-1}x^{(N-1)} + \ldots + a_0 x \tag{8}$$

under periodic conditions

$$\begin{aligned} x(0) &= x(2\pi) \\ x'(0) &= x'(2\pi) \\ &\cdots \\ x^{(N-1)}(0) &= x^{(N-1)}(2\pi) \end{aligned}$$

for continuous and bounded g.

We assume that L is a resonant operator, i.e. that the homogeneous problem $Lx = 0$ admits nontrivial periodic solutions. Namely, we assume that the polynomial

$$P(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + \ldots + a_0 \tag{9}$$

admits imaginary roots $\pm im$ $(m \in \mathbb{Z})$.

For notational convenience, let us introduce the n-dimensional symbolic vectors $V_{\pm\pm}$ given by

$$\begin{aligned} V_{++} &= (+\infty, +\infty, -\infty, -\infty, \ldots) \\ V_{+-} &= (+\infty, -\infty, -\infty, +\infty, \ldots) \\ V_{-+} &= (-\infty, +\infty, +\infty, -\infty, \ldots) \\ V_{--} &= (-\infty, -\infty, +\infty, +\infty, \ldots) \end{aligned}$$

where the sequences of signs are 4-periodic.

Then, the following result holds:

Theorem 2.2. *Let us assume that*

1. *The polynomial (9) has exactly two roots* $\pm im$ *in* $i\mathbb{Z}$, *which are simple.*
2. $g : \mathbb{R}^{N-1} \to \mathbb{R}$ *is a continuous bounded function such that the four limits*

$$\lim_{s \to V_{\pm\pm}} g(s) := g_{\pm\pm}$$

exist.

Let $p \in L^2(0,2\pi)$. If furthermore, we assume that

$$a_m^2(p) + b_m^2(p) < \frac{2}{\pi^2}\left[(g_{+-} - g_{-+})^2 + (g_{++} - g_{--})^2\right] \tag{10}$$

then equation (7) has at least one 2π-periodic solution in $H^N(0,2\pi)$.

3. Mawhin coincidence degree theory

One useful tool for proving this kind of results is Mawhin coincidence degree theory. Let us briefly summarize some aspects of this theory.

Let X and Y be real normed spaces, $L : \mathrm{dom}(L) \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index 0 if $\mathrm{Im}(L)$ is a closed subspace of Y and

$$\dim(\mathrm{Ker}(L)) = \mathrm{codim}(\mathrm{Im}(L)) < \infty$$

If L is a Fredholm mapping of index 0, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$ and $\mathrm{Ker}(Q) = Im(L)$. It follows that

$$L_P = L|_{\mathrm{dom}(L)\cap \mathrm{Ker}(P)} : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L) = \mathrm{Ker}(Q)$$

is one-to-one and onto $\mathrm{Im}(L)$. We denote its inverse by K_P. If Ω is a bounded open subset of X, N is called $L-$compact on Ω if $QN(\Omega)$ is bounded and $K_P(I - Q)N : \Omega \to X$ is compact. Since $\mathrm{Im}(Q)$ is isomorphic to $\mathrm{Ker}(L)$, there exists and isomorphism $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$.

The following continuation theorem is due to Mawhin [17]:

Theorem 3.1. *Let L be a Fredholm mapping of index zero and N be $L-$compact on Ω. Suppose*

1. *For each $\lambda \in [0,1]$, $x \in \partial\Omega$ we have that $Lx \neq \lambda Nx$*
2. *$QNx \neq 0$ for each $x \in Ker(L) \cap \partial\Omega$*
3. *The Brouwer degree satisfies: $d_B(JQN, \Omega \cap Ker(L), 0) \neq 0$*

Then the equation $Lx = Nx$ has at least one solution in $dom(L) \cap \Omega$.

This technique has been also applied to many other problems, see e.g. [6] and [11]. For further details see [17], [7].

4. Sketch of the proof of Theorem 2.1

In this section, we sketch the proof of theorem 2.1. For further details, see [1]. The proof of Theorem 2.2 is similar but somewhat more complicated (see [2]).

In this case, we shall consider $X = H^2_{per}(0,2\pi)$, $Y = L^2(0,2\pi)$ and L the linear differential operator

$$L(x) = x''' + ax'' + m^2x' + cx$$

It is immediate to see that $Ker(L) = \mathcal{E}_m$ is the subspace generated by $\sin(mt)$ and $\cos(mt)$ and that $\mathrm{Im}(L) = \mathcal{E}_m^{\perp}$. It follows that L is a Fredholm mapping of index zero. Moreover, we may take Q as the orthogonal projection P_m onto $\mathcal{E}_m$ in $L^2(0,2\pi)$ and P as the restriction of P_m to $H^2_{per}(0,2\pi)$. Ω will be an appropriate open bounded subset of $H^2_{per}(0, 2\pi)$. Then, if $Nx := p(t) - g(x')$, it can be proved that N is L-compact on Ω.

The proof requires three stages. First, we need an estimate for the linear operator L:

Lemma 4.1. $\|x - P_m(x)\|_{H^2} \leq c\,\|Lx\|_{L^2}\ \forall x \in H^3_{per}(0, 2\pi)$

Then, one needs to establish the following a priori bound:

Lemma 4.2. *Under the conditions of Theorem 2.1, there exist a constant C independent from $\lambda \in [0,1]$ such that if $x \in H^2(0, 2\pi)$ is a solution of*

$$L(x) = \lambda(p(t) - g(x'))$$

then $\|x\|_{H^2} \leq C$.

The proof follows by contradiction: if the result is not true, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of solutions such that $\|x_n\|_{H^2} \to +\infty$. We decompose x_n as

$$x_n = y_n + z_n$$

where $y_n = P_m(x_n)$ and $z_n = x_n - P_m(x_n) \in \mathcal{E}_m^{\perp}$. Using the previous estimate for the linear operator, some compactness arguments and a careful estimate of the oscillatory integrals that give the explicit expression of y_n , we get a contradiction.

It follows that if we take

$$\Omega = \{u \in H^2_{per}(0, 2\pi) : \|u\|_{H^2} < R\}$$

then there are no solutions $Lx = \lambda Nx$ on $\partial\Omega$. Hence, the coincidence degree $d(JQN, \Omega \cap \mathrm{Ker}L, 0)$ is well defined.

Finally, we need an explicit degree computation, taking into account the behavior of JQN restricted to the two dimensional kernel of L.

Lemma 4.3. *If*

$$a_m^2 + b_m^2 < \frac{4}{\pi^2}[g(+\infty) - g(-\infty)]^2$$

and R is large enough, then

$$d(JQN, \Omega \cap Ker(L), 0) = -1$$

This lemma involves also some estimates for the oscillatory integrals that give the explicit expression of the operator JQN, applied to functions in $\mathcal{E}_m$. Hence, all the conditions of the Mawhin continuation theorem are fulfilled, and the proof is complete.

References

[1] P. Amster, P. De Nápoli, M.C. Mariani. Periodic Solutions of a Resonant Third-Order Equation. To appear in Nonlinear Analysis, TMA.

[2] P. Amster, P. De Nápoli, M.C. Mariani. Periodic Solutions of a Resonant Higher Order Equation. Submitted.

[3] A. Caada, D. Ruiz. Resonant problems with multidimensional kernel and periodic nonlinearities, Differential and Integral Equations vol. 16, No. 4 (2003), pp. 499-512.

[4] J. Ezeilo, P. Omari. Nonresonant oscillations for some third-order differential equations. J. of Math. Anal. Appl 143 (1), (1989), pp. 56-56.

[5] J. Ezeilo, M. Nkashama. Resonant and nonresonant oscillations of some third order ordinary differential equations. Nonlinear Analysis, Theory and Applications 12 (10), (1988), pp. 1029-49.

[6] H. Fang, Z.C. Wang. Periodic Boundary Value Problems of Impulsive Differential Equations. Applied Mathematics E-Notes, 1 (2001), pp. 77–85.

[7] R. Gaines, J. Mawhin, Lecture Notes in Mathematics, Vol 586, Springer-Verlag, Berlin, (1997).

[8] M. Grossinho and T.F.Ma: Symmetric equilibria for a beam with a nonlinear elastic foundation. Portugaliae Mathematica, 51 (1994), pp. 375-393

[9] M. Grossinho and F.Minhós: Existence Results for some third order separated boundary value problems. Nonlinear Analysis 47 (2001), pp. 2407-2418.

[10] M. Grossinho and S.Tersian: The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation. To appear in Nonlinear Analysis, Theory, Methods, and Applications.

[11] S. Iyase. Non-resonant oscillations for some fourth-order differential equations with delay. Math. Proc. of the Royal Irish Academy, 99A (1) 1999,pp. 113-121.

[12] E. Landesman, A. Lazer. Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), pp. 609–623.

[13] A. Lazer. A Second look at the first result of Landesman-Lazer type. Electron. J. Diff. Eqns., Conf. 05, 2000, pp. 113–119.

[14] A. Lazer, On Schauder's fixed point theorem and forced second order nonlinear oscillations, J. Math. Anal. Appl. 21(1968), pp. 421-425.

[15] A. Lazer, D. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl. 82 (1969), pp. 49-68.

[16] R. Leuchtag, Family of differential equations arising from multi-ion electrodiffussion, J. Math. Phys. 22 (6) 1981, 1317-1320.

[17] J. Mawhin. Topological degree methods in nonlinear boundary value problems, NSF-CBMS Regional Conference in Mathematics no. 40, American Mathematical Society, Providence, RI, (1979).

[18] J. Mawhin. Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance. Bol. de la Sociedad Española de Mat.Aplicada 16 (2000), 45-65.

[19] J. Mawhin. Seventy-five years of global analysis around the pendulum equation. Proc. Equadiff 9, Brno (1997).

[20] F. Minhs. Periodic solutions for a third order differential equation under conditions on the potential. Portugalie Mathematica, Vol. 55 Fasc. 4, (1998).

[21] R. Ortega, L.Sanchez. Periodic Solutions of a forced oscillator with several degrees of freedom. Bull London Math Soc. Vol 34.

Pablo Amster (*) (e-mail:pamster@dm.uba.ar)
Pablo L. De Nápoli (e-mail: pdenapo@dm.uba.ar)
M. Cristina Mariani (*) (e-mail: mcmarian@dm.uba.ar)
Universidad de Buenos Aires
FCEyN - Departamento de Matemática
Ciudad Universitaria, Pabellón I
(1428) Buenos Aires, Argentina
(*) Conicet