ACTAS DEL VII CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2003) , Páginas 1–9

A DISTRIBUTIONAL CONVOLUTION PRODUCT OF $(c^2 + P \pm i0)^\lambda$

MANUEL A. AGUIRRE T.

ABSTRACT. In this article we obtain two results: first we find a relation between the generalized functions $(c^2+P\pm i0)^\lambda$and the distribution $\delta^{(k)}(P\pm i0+c^2)$((16), as consequence we obtain a expansion style Taylor's series of $\delta^{(k)}(P\pm i0+c^2)$ see (26)) and then we give a sense to certain convolution of the form $(c^2+P\pm i0)^\lambda*(c^2+P\pm i0)^\mu$(44), where $(c^2+P\pm i0)^\lambda$is defined by(2). Our formula(44) is a generalization of the convolution $(P\pm i0)^\lambda*(P\pm i0)^\mu$(56),this formula was proved by S.E.Trione in([6],page 40) and ([10],pages 51-62). The interest of this study is due to the fact that many of the divergences appearing in quantum electrodynamics arise from divergent convolution $(P\pm i0)^\lambda*(P\pm i0)^\mu$.

1. INTRODUCTION.

Let P be a quadratic form in n variables defined by

$$P = P(x) = x_1^2 + x_2^2 + ...x_p^2 - x_{p+1}^2 - ...x_{p+q}^2 \tag{1}$$

where $n = p + q$(dimension of the space).

The distributions $(c^2 + P \pm i0)^\lambda$ are defined by

$$(c^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to o}(c^2 + P \pm i\varepsilon)^\lambda \tag{2}$$

([1], page 289), where $\varepsilon > o$, c a positive real, λ is a complex number and

$$|x|^2 = x_1^2 + x_2^2 + ...x_n^2.$$

Similarly the distributions $(P \pm i0)^\lambda$ are defined defined by the following formula

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to o}(P \pm i\varepsilon)^\lambda \tag{3}$$

([1], page 275).

The distributions $(P\pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2}-k, k = 0, 1, 2, ...$ where they have simple poles([1], page 275).

From([2], pag116,formulae30,31,and32) we have,

If n is odd,

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k}(P \pm i0)^\lambda = \frac{e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L^k\{\delta(x)\} \tag{4}$$

Key words and phrases. Theory of Distributions.

2000 *Mathematics Subject Classification.* 46F10, 46F12.

Work partially support by Comisión de Investigaciones Científicas(C.I.C.)(Argentina).

where L^k is the $k-th$ iteration of the differential operator of the form

$$L^k = \left\{ \frac{\partial^2}{\partial^2 x_1} + \ldots + \frac{\partial^2}{\partial^2 x_p} - \frac{\partial^2}{\partial^2 x_{p+1}} - \ldots \frac{\partial^2}{\partial^2 x_{p+q}} \right\}^k . \tag{5}$$

If n is even:
a) p and q both are even

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k} (P \pm i0)^\lambda = \frac{e^{\mp \frac{q\pi i}{2}} \pi^{\frac{n}{2}}}{2^{2k} k! \Gamma(\frac{n}{2}+k)} L^k \{\delta(x)\}, \tag{6}$$

b) p and q both are odd

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k} (P \pm i0)^\lambda = 0 \tag{7}$$

Also we may note from([1], page 278) that if n is odd, and also if n is even and $k < \frac{n}{2}$ we have,

$$\operatorname{Re}s_{\lambda=-k,k=1,2,\ldots} (P \pm i0)^\lambda = 0. \tag{8}$$

On the other hand, from([2], page 121, formula 66) we have the following formula

$$(c^2 + P \pm i0)^\lambda = \sum_{\nu \geq o} \binom{\lambda}{\nu} (c^2)^\nu (P \pm i0)^{\lambda-\nu} \tag{9}$$

where

$$\binom{\lambda}{\nu} = \frac{\Gamma(\lambda+1)}{\nu!\Gamma(\lambda-\nu+1)} = \frac{(-1)^\nu \Gamma(-\lambda+\nu)}{\nu!\Gamma(-\lambda)}. \tag{10}$$

In this article we obtain two results: first we find a relation between the generalized functions $(c^2 + P \pm i0)^\lambda$and the distribution $\delta^{(k)}(P \pm i0 + c^2)$((16), as consequence we obtain a expansion style Taylor's series of $\delta^{(k)}(P \pm i0 + c^2)$ see (26)) and then we give a sense to certain convolution of the form $(c^2+P\pm i0)^\lambda * (c^2+P\pm i0)^\mu$(44),where $(c^2+P\pm i0)^\lambda$is defined by(2). Our formula(44) is a generalization of the convolution $(P \pm i0)^\lambda * (P \pm i0)^\mu$(56),this formula is was proved by S.E.Trione in([6],page 40) and([10],page 51-62) . The interest of this study is due to the fact that many of the divergences appearing in quantum electrodynamics arise from divergent convolution $(P \pm i0)^\lambda * (P \pm i0)^\mu$.

Now we will study the singularities of distributions $(c^2 + P \pm i0)^\lambda$

2. The residue of $(c^2 + P \pm i0)^\lambda$

We observe that the distributions $(P \pm i0)^\lambda$ have singularities at $\lambda = -\frac{n}{2} - k, k = 0, 1, 2, \ldots$,therefore from(9) the distributions $(c^2 + P \pm i0)^\lambda$ have singularities at $\lambda =$

$-\frac{n}{2}-k, k=0,1,2,...,$ and using (4),(6),(7) and (10) we have

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k}(c^2+P\pm i0)^\lambda =$$

$$\textstyle\sum_{\nu\geq o}\frac{(-1)^\nu\Gamma(\frac{n}{2}+k+\nu)}{\nu!\Gamma(\frac{n}{2}+k)}(c^2)^\nu \operatorname{Re}s_{\alpha=-\frac{n}{2}-(k+\nu)}(P\pm i0)^\alpha = \tag{11}$$

$$\textstyle\frac{e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+k)2^{2k}}\sum_{\nu\geq o}\frac{(-1)^\nu}{\nu!(k+\nu)!2^{2\nu}}(c^2)^\nu L^{k+\nu}\{\delta(x)\}$$

if n is odd. When n is even we have the following cases

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k}(c^2+P\pm i0)^\lambda = \frac{e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+k)2^{2k}}\sum_{\nu\geq o}\frac{(-1)^\nu}{\nu!(k+\nu)!2^{2\nu}}(c^2)^\nu L^{k+\nu}\{\delta(x)\} \tag{12}$$

if p and q are both even,

$$\operatorname{Re}s_{\lambda=-\frac{n}{2}-k}(c^2+P\pm i0)^\lambda = 0 \tag{13}$$

if p and q are both odd, and

$$\operatorname{Re}s_{\lambda=-l,l=1,2,...}(c^2+P\pm i0)^\lambda = 0 \tag{14}$$

if $l<\frac{n}{2}$.

In the next section we will study the residue of $(c^2+P\pm i0)^\lambda$ at $\lambda=-k, k=1,2,...,$ under condition $k\geq\frac{n}{2}$.

3. The residue of $(c^2+P\pm i0)^\lambda$ at $\lambda=-k, k=1,2,...,$ under condition $k\geq\frac{n}{2}$.

To obtain the residue of $(c^2+P\pm i0)^\lambda$ at $\lambda=-k, k=1,2,...,$ under condition $k\geq\frac{n}{2}$,with n even we need the following formula

$$\delta^{(k)}(P\pm i0-m^2) = \frac{(-1)^k e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}}\sum_{\nu\geq o}\frac{(m^2)^\nu}{4^\nu\nu!(\nu-\frac{n}{2}+k+1)!}L^{\nu-\frac{n}{2}+k+1}\{\delta(x)\} \tag{15}$$

if n is even and $k\geq\frac{n}{2}$([3], page 344, formula 4.1).

Theorem 1. *Let $(c^2+P\pm i0)^\lambda$ the distribution defined by (2) and the distribution $\delta^{(k)}(P\pm i0-m^2)$ defined by (15) then the following formula is valid*

$$Res_{\lambda=-k-1}(c^2+P\pm i0)^\lambda = \frac{(-1)^k}{k!}\delta^{(k)}(P\pm i0+c^2) \tag{16}$$

under conditions $k\geq\frac{n}{2}$ and n even.

Proof. Putting $s=k-\frac{n}{2}+1$ and $c^2=-m^2$ in (15) we have,

$$\delta^{(k)}(P\pm i0+c^2)= \\ =\frac{(-1)^{s-\frac{n}{2}+1}e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{4^s}\sum_{\nu\geq o}\frac{(-c^2)^\nu}{4^\nu\nu!(\nu+s)!}L^{\nu+s}\left\{\delta(x)\right\} \tag{17}$$

On the other hand, from (12) we have,

$$\underset{\lambda=-\frac{n}{2}-s}{Res}(c^2+P\pm i0)^\lambda= \\ =\frac{e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+s)4^s}\sum_{\nu\geq o}\frac{(-c^2)^\nu}{4^\nu\nu!(\nu+s)!}L^{\nu+s}\left\{\delta(x)\right\}. \tag{18}$$

From (17) and (18) we have,

$$\underset{\lambda=-\frac{n}{2}-s}{Res}(c^2+P\pm i0)^\lambda=\frac{(-1)^{s+\frac{n}{2}-1}}{\Gamma(\frac{n}{2}+s)}\delta^{(k)}(P\pm i0+c^2) \tag{19}$$

Taking into account that n is even and using that

$$\Gamma(\frac{n}{2}+s)=(\frac{n}{2}+s-1)$$

from (19) we have,

$$\underset{\lambda=-\frac{n}{2}-s}{Res}(c^2+P\pm i0)^\lambda=\frac{(-1)^{s+\frac{n}{2}-1}}{(\frac{n}{2}+s-1)!}\delta^{(k)}(P\pm i0+c^2) \tag{20}$$

From (20) and using that

$$s=k-\frac{n}{2}+1 \tag{21}$$

we obtain the following formula

$$\underset{\lambda=-k-1}{Res}(c^2+P\pm i0)^\lambda=\frac{(-1)^k}{k!}\delta^{(k)}(P\pm i0+c^2) \tag{22}$$

which coincide with the formula (16). □

It's clear that putting $c^2=0$ in both terms of the formula (22) we obtain the formula

$$\underset{\lambda=-k-1}{Res}(P\pm i0)^\lambda=\frac{(-1)^k}{k!}\delta^{(k)}(P\pm i0) \tag{23}$$

where

$$\delta^{(k)}(P\pm i0)=\frac{(-1)^k e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} \tag{24}$$

The formulae (23) and (24)) appear in ([4], page 39,formula 62).

On the other hand from (18) and using (21) and (24) we obtain the following formula

$$\underset{\lambda=-k-1}{Res}(c^2+P\pm i0)^\lambda=\frac{(-1)^k}{k!}\sum_{\nu\geq o}\frac{(c^2)^\nu}{\nu!}\delta^{(k+\nu)}(P\pm i0) \tag{25}$$

From (22) and (25) we obtain the following expansion style Taylor's series

$$\delta^{(k)}(P \pm i0 + c^2) = \sum_{\nu \geq o} \frac{(c^2)^\nu}{\nu!} \delta^{(k+\nu)}(P \pm i0). \tag{26}$$

4. The convolution product of $(c^2 + P \pm i0)^\lambda * (c^2 + P \pm i0)^\mu$

We shall now evaluate the convolution product of

$$(c^2 + P \pm i0)^\lambda * (c^2 + P \pm i0)^\mu$$

taking into account the formula (9) and the Fourier transform of distribution $(P \pm i0)^\lambda$.

From ([1], page 284,formulae 3 and 3',we have,

$$F\left\{(P \pm i0)^\lambda\right\} = a(\lambda, q, n)(Q \mp i0)^{-\lambda - \frac{n}{2}} \tag{27}$$

where

$$Q = Q(y) = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2, \tag{28}$$

$$(Q \mp i0)^{-\lambda - \frac{n}{2}}$$

is defined by the formula (3), $F\left\{(P \pm i0)^\lambda\right\}$indicates the Fourier transform of the distribution $(P \pm i0)^\lambda$:

$$F\{f_\alpha\} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{i(x,y)} f_\alpha(x) dx \tag{29}$$

and

$$a(\lambda, q, n) = \frac{e^{\mp \frac{q\pi i}{2}} 2^{2\lambda + n} \pi^{\frac{n}{2}} \Gamma(\lambda + \frac{n}{2})}{\Gamma(-\lambda)(2\pi)^{\frac{n}{2}}}. \tag{30}$$

Using the formula (9) we give a sense the Fourier transform of $(c^2 + P \pm i0)^\lambda$:

$$F\left\{(c^2 + P \pm i0)^\lambda\right\} = \sum_{\nu \geq o} \binom{\lambda}{\nu} (c^2)^\nu F\left\{(P \pm i0)^{\lambda - \nu}\right\}. \tag{31}$$

From ([6],pages23-25 and ([11],pages315-326)) and ([1], page 285) the last formula can be justified by the following form:

Let $f(z, \lambda)$, $z \in C$, be an entire function of the variables z, λ :$f(z, \lambda) = \sum_{\nu \geq o} b_\nu(\lambda) z^\nu$.

Let us consider the family of distribution of the form ([1],page 285)

$$T(P \pm i0, \lambda) = (P \pm i0)^\lambda f((P \pm i0, \lambda) = (P \pm i0)^\lambda \sum_{\nu \geq o} b_\nu(\lambda)(P \pm i0)^\nu. \tag{32}$$

To evaluate the Fourier transform of $T(P \pm i0, \lambda)$ in sense of Gelfand(c.f.[1]) we have to show that

$$F\left\{(P \pm i0)^\lambda \sum_{\nu \geq o} b_\nu(\lambda)(P \pm i0)^\nu\right\} = \sum_{\nu \geq o} b_\nu(\lambda) F\left\{(P \pm i0)^{\lambda + \nu}\right\}. \tag{33}$$

Let us suppose, provisionally that $Re(\lambda) > -1$; then the terms of the sequence $(n = 0, 1, 2, ...)$

$$\{g_n\} = \left\{ (P \pm i0)^{\lambda} \sum_{\nu=o}^{n} b_{\nu}(\lambda)(P \pm i0)^{\nu} \right\} \tag{34}$$

are locally integrable functions. Since, by hypothesis, $f(z, \lambda)$ is an entire function, we conclude that the sequence (34) converges uniformly in every compact $K \subset R^n$.

Therefore, by([7], theorem XVI, page 76), the sequence $\{g_n\}$ is convergent in D',and by the continuity of the Fourier transform, we conclude that the equation (33) is valid when $Re(\lambda) > -1$.

Then, taking into account the formulae (10),(27) and(30) we have

$$F\left\{(c^2 + P \pm i0)^{\lambda}\right\} = \sum_{\nu \geq o} \binom{\lambda}{\mu} (c^2)^{\nu} a(\lambda - \nu, q, n)(Q \mp i0)^{-\lambda - \frac{n}{2} + \nu} \tag{35}$$

if $\lambda \neq -\frac{n}{2} \pm k, k = 0, 1, 2, ...,$ n dimension of the space,where $a(\lambda - \nu, q, n)$ is defined by (30).

On the other hand, from([7]), theorem XV, page 268, the Fourier´s transform F and $\overline{F}$ are reciprocal isomorphisms from the space O_M and the space $O_c^{'}$ respectively. In addition

$$\text{if } T \in O_M \Longrightarrow \overline{F}[T] \in O_c^{'} \tag{36}$$

and

$$\text{if } T \in O_c^{'} \Longrightarrow \overline{F}[T] \in O_M \tag{37}$$

where $O_c^{'}$ is the space of rapidly decreasing distribution([7],page 244) and O_M is the space of all infinitely differentiable functions such that they and their derivatives are slow growth([7],page243) and if

$$g = F[T] \Longrightarrow f = \overline{F}[T] = F^{-1}[g]\,. \tag{38}$$

Now using that $(1 + r^2)^{-m} \in O_M$([7],page 243) where $r^2 = x_1^2 + ... + x_p^2 + x_{p+1}^2 + ...x_{p+q}^2$, from ([5]) we obtain that

$$(Q \pm i0)^{-\frac{\alpha}{2}} \in O_M \tag{39}$$

for all α complex number such that $\frac{\alpha}{2} \neq \frac{n}{2} + k, k = 0, 1, ...$

From (35)and using(39) we have

$$F\left\{(c^2 + P \pm i0)^{\lambda}\right\} \in O_M \tag{40}$$

if $\lambda \neq -\frac{n}{2} \pm k, k = 0, 1, 2, ...,$ n dimension of the space.

Now from(40) and using the propertie(36) and (38) we have

$$(c^2 + P \pm i0)^{\lambda} \in O_c^{'} \tag{41}$$

if $\lambda \neq -\frac{n}{2} - k, k = 0, 1, 2, ...,$ n dimension of the space and taking into account that $O_M, O_c^{'} \subset S'$, where S' is the dual of S and S is the Schwartz clas of infinitely differential functions on R^n decreasing and infiny faster that $\mid x \mid^{-1}$ ([7],page234), we have

$$(c^2 + P \pm i0)^{\lambda} \in S^{'}. \tag{42}$$

if $\lambda \neq -\frac{n}{2} - k, k = 0, 1, 2, ...,$ n dimension of the space.

From(41) and (42) and appealing to and considering the classical theorem of Schwartz ([7],page268,), we define the distributional convolution product of $(c^2 + P \pm i0)^\lambda$ and $(c^2 + P \pm i0)^\mu$ by means of the following :

Definition 2. *Let λ and μ be complex numbers such that λ, μ and $\lambda + \mu \neq -\frac{n}{2} + r, r = 0, 1, 2, ...$ and n dimension of the space, we define the distributional convolution product between $(c^2 + P \pm i0)^\lambda$ and $(c^2 + P \pm i0)^\mu$ by the formula*

$$(c^2+P\pm i0)^\lambda * (c^2+P\pm i0)^\mu = (2\pi)^{\frac{n}{2}} F^{-1} \left\{ F \left[((c^2 + P \pm i0)^\lambda . F \left[((c^2 + P \pm i0)^\mu \right] \right\} \quad (43)$$

where F^{-1} means the inverse of the Fourier's transform in the sense of the formula(38).

Using the above definition we get:

Theorem 3. *Let λ and μ be complex numbers such that λ, μ and $\lambda+\mu \neq -\frac{n}{2}+r, r = 0, 1, 2, ...$ and n dimension of the space then the distributional convolution product between $(c^2 + P \pm i0)^\lambda$ and $(c^2 + P \pm i0)^\mu$ define by (43) can be written in the following form*

$$(c^2 + P \pm i0)^\lambda * (c^2 + P \pm i0)^\mu =$$

$$e^{\mp \frac{q\pi i}{2}} \pi^{\frac{n}{2}} A_{\lambda,\mu,n} \sum_{l \geq o} \binom{\lambda+\mu+\frac{n}{2}}{l} \left\{ B(\lambda + \tfrac{n}{2} - l, \mu + \tfrac{n}{2} - l) \right. \quad (44)$$

$$\left. (c^2)^l (P \pm i0)^{\lambda+\mu+\frac{n}{2}-l} \right\}$$

where

$$A_{\lambda,\mu,n} = \frac{\Gamma(-\lambda - \mu - \frac{n}{2})}{\Gamma(-\lambda)\Gamma(-\mu)} \quad (45)$$

and

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \quad (46)$$

for Re$(x) > 0$, Re$(y) > 0$ ([8],page9,formula5).

Proof. From(36),using(27),(35) and taking into account the formula

$$(Q \mp i0)^\lambda . (Q \mp i0)^\mu = (Q \mp i0)^{\lambda+\mu} \quad (47)$$

([6],page23,formula(I,3,1)) where $\lambda, \mu \in C$ and λ, μ and $\lambda - \mu$ are different from $-\frac{n}{2} - k, k = 0, 1, 2, ...,$ we have,

$$F \left[(c^2 + P \pm i0)^\lambda \right] . F \left[(c^2 + P \pm i0)^\mu \right] =$$

$$(2\pi)^{\frac{n}{2}} \sum_{l \geq o} R_l(\lambda, \mu, q, n)(c^2)^l (Q \mp i0)^{-(\lambda+\mu+\frac{n}{2}+l)-\frac{n}{2}} \quad (48)$$

if λ, μ and $\lambda + \mu \neq -\frac{n}{2} \pm r, r = 0, 1, 2, ...$ where

$$R_l(\lambda, \mu, q, n) = \sum_{\nu=o}^{l} \binom{\lambda}{\nu} \binom{\mu}{l - \nu} (c^2)^l a(\lambda - \nu, q, n) a(\mu - (l - \nu), q, n) \quad (49)$$

and $a(\gamma,q,n)$ is define by the formula(30).

Now using the propertie(27) and (33),from(48) we have

$$F^{-1}\left\{F\left[(c^2+P\pm i0)^{\lambda}\right].F\left[(c^2+P\pm i0)^{\mu}\right]\right\}= \\ (2\pi)^{\frac{n}{2}}\sum_{l\geq o}T_l(\lambda,\mu,q,n)(c^2)^l(P\pm i0)^{\lambda+\mu+\frac{n}{2}-l} \tag{50}$$

if λ,μ and $\lambda+\mu\neq-\frac{n}{2}+r, r=0,1,2,...,$ n dimension of the space. Where

$$T_l(\lambda,\mu,q,n)=\frac{R_l(\lambda,\mu,q,n)}{a(\lambda+\mu+\frac{n}{2}-l,q,n)}. \tag{51}$$

From(51),using the formulae(10),(30),(49)and taking into account the formulae

$$\frac{\Gamma(z)}{\Gamma(z-s)}=\frac{(-1)^s\Gamma(-z+s+1)}{\Gamma(1-z)} \tag{52}$$

([8],page 344),

$$\sum_{\nu=o}^{l}\frac{l!}{\nu!(l-\nu)!\Gamma(-\lambda-\frac{n}{2}+\nu+1)\Gamma(-\mu-\frac{n}{2}+(l-\nu)+1)}= \\ =\frac{\Gamma(-\lambda-\frac{n}{2}-\mu-\frac{n}{2}+2l+1)}{\Gamma(-\lambda-\frac{n}{2}+l+1)\Gamma(-\mu-\frac{n}{2}+l+1)\Gamma(-\lambda-\frac{n}{2}-\mu-\frac{n}{2}+l+1)} \tag{53}$$

([9],page147) and after long but elementary calculation we get

$$T_l(\lambda,\mu,q,n)=\frac{e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}(-1)^l}{(2\pi)^{\frac{n}{2}}}.\frac{\Gamma(-\lambda-\mu-\frac{n}{2})}{\Gamma(-\lambda)\Gamma(-\mu)}. \\ \binom{\lambda+\mu+\frac{n}{2}}{l}B(\lambda+\frac{n}{2}-l,\mu+\frac{n}{2}-l). \tag{54}$$

From(50) and using(54) we obtain the formula(44)and finished the proof of theorem. □

On the other hand, letting $\lambda=\frac{\alpha-n}{2}$ and $\mu=\frac{\beta-n}{2}$ in(44)we obtain the following formula

$$(c^2+P\pm i0)^{\frac{\alpha-n}{2}}*(c^2+P\pm i0)^{\frac{\beta-n}{2}}= \\ e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2}+\frac{n-\beta}{2}-\frac{n}{2})}{\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{n-\beta}{2})}\sum_{l\geq o}\binom{\frac{\alpha-n}{2}+\frac{\beta-n}{2}+\frac{n}{2}}{l}B(\frac{\alpha}{2}-l,\frac{\beta}{2}-l) \\ (m^2)^l(P\pm i0)^{\frac{\alpha-n}{2}+\frac{\beta-n}{2}+\frac{n}{2}-l}\Big\} \tag{55}$$

where α,β and $\alpha+\beta\neq n+2r, r=0,1,2,...$ and $B(\frac{\alpha}{2}-l,\frac{\beta}{2}-l)$ is given by the formula(46).

In particular taking $c^2=0$ in(55)and using the formula(46)we obtain the following convolution distributional product

$$(P\pm i0)^{\frac{\alpha-n}{2}}*(P\pm i0)^{\frac{\beta-n}{2}}=e^{\mp\frac{q\pi i}{2}}\pi^{\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2}+\frac{n-\beta}{2}-\frac{n}{2})}{\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{n-\beta}{2})} \\ \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})}(P\pm i0)^{\frac{\alpha+\beta-n}{2}} \tag{56}$$

under conditions α, β and $\alpha + \beta \neq n + 2r, r = 0, 1, 2, ...$

The formula(56) has been proved by S.E.Trione in([6], page 40,formula(II,3,11) and([10],page 51-62).

REFERENCES

[1] I.M.Gelfand and G.E.Shilov, Generalized Function, Vol. I, Academic Press, New York, 1964

[2] M.Aguirre T., The Expansion and Fourier's Transforma of $\delta^{(k-1)}(m^2 + P)$,Integral Transform and Special Functions, Vol.3, Nro.2, pp.113-134, 1995.

[3] M.Aguirre T., The distributional $\delta^{(k)}(P \pm i0 - m^2)$, Journal of Computational and Applied Mathematics 88 (1997), 339-348.

[4] M.A.Aguirre T. and A.L.Barrenechea., A relation between the kth derivative of Dirac delta in $(P \pm i0)$ and the residue of distributions $(P \pm i0)^{\lambda}$,Journal of Computational and Applied Mathematics 108 (1999),31-40.

[5] M.A.Aguirre T. and A.Kananthai., On the convolution product of the Distributional Families related to the Diamond Operatyor, Le Matematiche, vol.56, fasc.2, 2003.

[6] S.E.Trione., Distributional Products, Cursos de Matemática, N°3, Serie II, IAM-CONICET,1980.

[7] L. Schwartz., Théorie des distributions, Herman, Paris, 1966

[8] A. Erdelyi(Ed.), Higher Trascendental Functions, Vol.I and II, McGraw-Hill, New York, 1953.

[9] G.N.Watson., A treatisc on the theory of Bessel functions,second edition,Cambrige,University Press,1994.

[10] A.González Dominguez and S.E.Trione., On the Laplace transform of retarded invariant functions, Advances in Mathematics,Volume 30,number 2,1978,pp.51-62.

[11] S.E.Trione., On the Fourier transform of causal distributions, Studie in Applied Mathematics,Massachusetts Institute of Tecnology, Cambrige, USA, 55,pp.315-326, 1976.

NÚCLEO CONSOLIDADO MATEMÁTICA PURA Y APLICADA, FACULTAD DE CIENCIAS EXACTAS, UNIVERSIDAD NACIONAL DEL CENTRO, PINTO 399,(7000)TANDIL, ARGENTINA

E-mail address: maguirre@@exa.unicen.edu.ar