A Note on the Equational Bases for Subvarieties of Linear Symmetric Heyting Algebras

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Abstract

In this paper, the construction of the lattice of subvarieties of linear symmetric Heyting algebras is given and equational bases for each subvariety are obtained.

1 Introduction

A symmetric Heyting algebra is an algebra $\langle L, \wedge, \vee, \Rightarrow, \sim, 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ such that $\langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ is a Heyting algebra and $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra (see [9]).

The study of the variety of symmetric Heyting algebras was pioneered by A. Monteiro. It was pursued by, among others, H. Sankappanavar [10] and L. Iturrioz [6]. In [6], L. Iturrioz gave a complete description of the variety $\underset{\sim}{D}_3$ of three-valued symmetric Heyting algebras. Later on, A. Monteiro comprehensively investigated the variety of symmetric Heyting algebras and several of its subvarieties in his very important work "Sur les algèbres de Heyting symétriques" [9]. Particularly, he studied the subvariety of linear symmetric Heyting algebras, that is, symmetric Heyting algebras satisfying the identity

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1.$$

These algebras reflect algebraically the properties of modal symmetric propositional calculus in the same way as Heyting algebras are algebraic structures imposed by the study of intuitionistic proposicional calculus.

Linear symmetric Heyting algebras form an equational class $\mathcal{L}$. In this paper we consider the lattice of subvarieties of this variety. We describe the structure of the poset of its join-irreducible elements and we find equational bases for each subvariety of $\mathcal{L}$.

We draw heavily on Monteiro's paper [9].

Linear symmetric Heyting algebras can be characterized by the condition that the poset of filters containing a prime filter is a chain [9].

The importance of the following examples of linear symmetric Heyting algebras will be clear later.

Let C_n, $n \geq 2$, be the Heyting algebra of all fractions $\frac{i}{n-1}$, $i = 0, 1, \ldots, n-1$ ([9], p. 136), with $\sim x = 1 - x$, and let D_n be the Heyting algebra $C_n \times C_n$, with $\sim (x, y) = (1 - y, 1 - x)$. C_n and D_n are linear symmetric Heyting algebras.

An I_n-algebra is a symmetric Heyting algebra satisfying the Ivo Thomas identity:

$$\gamma_n(x_0, x_1, \ldots, x_{n-1}) = \beta_{n-2} \Rightarrow (\beta_{n-3} \Rightarrow (\ldots \Rightarrow (\beta_0 \Rightarrow x_0) \ \ldots)) = 1,$$

where $\beta_i = (x_i \Rightarrow x_{i+1}) \Rightarrow x_0$ for $i = 0, 1, \ldots, n-2$ (see [9], p. 136, and [12]).

The algebras C_n and D_n are examples of symmetric Heyting algebras that satisfy the identity $\gamma_n = 1$.

The following result is a characterization of the I_n-algebras.

Lema 1.1 [9] *For a linear Heyting algebra A, the following are equivalent:*

1. *The identity $\gamma_n = 1$ holds.*
2. *The poset of proper prime filters containing a prime filter P is a chain of length at most $n - 1$.*

A. Monteiro proved ([9], p. 138, Th.1.6) that the variety $\underset{\sim}{I}_n$ of I_n-algebras is generated in $\mathcal{L}$ by $D_n \times D_{n-1}$.

It is clear that a finite linear symmetric algebra is an I_n-algebra for some n. Then we have the following Theorem:

Theorem 1.2 *If A is a finite algebra in $\mathcal{L}$, then A is subdirectly irreducible if and only if there exists n such that either A is isomorphic to D_n or A is isomorphic to C_n.*

If G is a finite subset of an algebra $A \in \mathcal{L}$, then the subalgebra generated by G is the Heyting subalgebra generated by $G \cup \sim G$. Since the variety of linear Heyting algebras is locally finite [1], it follows that $\mathcal{L}$ is locally finite. In addition, $\mathcal{L}$ has the congruence-distributive property, being that the lattice of congruences in an algebra A is a sublattice of the lattice of congruences of the Heyting algebra A, and the latter is congruence-distributive.

We conclude this section by recalling the characterization of subalgebras of the algebras C_n and D_n [11].

Let $n \geq 2$. If n is even, then the subalgebras of C_n are the algebras C_{2k}, $k \leq n/2$. If n is odd, then C_k is a subalgebra of C_n for every $k \leq n$.

Let $S_Y = C_n - Y$, where $Y \subseteq C_n - \{0, 1\}$. Let $S_{\mathcal{H}}$ be the set of Heyting subalgebras of C_n. Then $S_{\mathcal{H}} = \{S_Y : Y \subseteq C_n - \{0, 1\}\}$. For every j, $2 \leq j \leq n$, let $Y \in S_{\mathcal{H}}$ be such that $|Y| = n - j$. Then $A = S_Y \times S_{\sim Y}$ is a subalgebra of D_n isomorphic to D_j. In addition, $D_i \subseteq D_j$ if and only if $i \leq j$. If A is a subalgebra of D_n and A is not isomorphic to D_k, for any k, then $A \simeq C_t$, for $t \leq n$. We have that $A = \{(x, \alpha(x)), x \in p_1(A)\}$, where α is an isomorphism from $p_1(A)$ onto $p_2(A)$, p_1, p_2 the projections in $D_n = C_n \times C_n$.

2 Subvarieties

Given a class K of algebras, let $\mathbf{Si}(K)$ and $\mathbf{Si}_{\text{fin}}(K)$ consist of one representative from each isomorphism class of subdirectly irreducible and finite subdirectly irreducible, respectively, algebras in K.

Since $\mathcal{L}$ is congruence-distributive, we can apply a theorem of Jónsson [8] and its generalization (see Davey [5]) to find the lattice $\Lambda(\mathcal{L})$ of subvarieties of $\mathcal{L}$. Jónsson's Theorem may be stated as follows:

Theorem 2.1 [8] *Let $V = V(K)$ a congruence-distributive variety that is generated by a finite set K of finite algebras and order $\mathbf{Si}(V)$ by:*

$$A \leq B \Leftrightarrow A \in \mathbf{H}(\mathbf{S}(B)).$$

Then the lattice $\Lambda(V)$ of subvarieties of V is a finite distributive lattice which is isomorphic to $\mathcal{O}(\mathbf{Si}(V))$, the lattice of down-sets (order-ideals) of the ordered set $\mathbf{Si}(V)$. Moreover, a subvariety $X \in \Lambda(V)$ is join-irreducible if and only if $X = V(A)$, for some subdirectly irreducible algebra A.

Davey proved the following generalization:

Theorem 2.2 [5] *Let V a locally finite congruence-distributive variety. Then $\Lambda(V)$ is a completely distributive lattice and is isomorphic to $\mathcal{O}(\mathbf{Si}_{\text{fin}}(V))$.*

The order in $\mathbf{Si}(\mathcal{L})$ (and in $\mathbf{Si}_{\text{fin}}(\mathcal{L})$) is the following:

$$A \leq B \text{ if and only if } A \in \mathbf{S}(B)$$

being that if $A \in \mathbf{Si}(\mathcal{L})$ then A is simple, that is, the unique homomorphic images are the trivial ones.

Let $\underset{\sim}{D}_n$ and $\underset{\sim}{C}_n$ denote the varieties generated by D_n and C_n, respectively, that is, $\underset{\sim}{D}_n = V(D_n)$ and $\underset{\sim}{C}_n = V(C_n)$, and for a distributive lattice R, $\mathcal{J}(R)$ denotes the ordered set of all join-irreducible elements of the distributive lattice R.

Let $\underset{\sim}{K} = V(\bigcup_{n\geq 2} C_n)$. This is the variety called by A. Monteiro the variety of totally linear symmetric Heyting algebras.

Let $\underset{\sim}{P} = V(\bigcup_{n\geq 1} C_{2n})$.

It is clear that $\underset{\sim}{P} \subseteq \underset{\sim}{K}$. Furthermore, $\underset{\sim}{P} \neq \underset{\sim}{K}$. Indeed, for $A \in \mathbf{Si}_{\text{fin}}(\underset{\sim}{P})$, A is isomorphic to C_{2n}, and then, for odd t, there isn't $A \in \mathbf{Si}_{\text{fin}}(\underset{\sim}{P})$, such that $C_t \in \mathbf{S}(A)$. Thus, $\underset{\sim}{P} \subset \underset{\sim}{K}$.

Therefore, $\mathcal{J}(\Lambda(\mathcal{L}))$ is the poset indicated in fig. 1.

Theorem 2.3 *$\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathcal{L}$ are join-irreducible in $\Lambda(\mathcal{L})$.*

Proof. Suppose $\underset{\sim}{K} = V_1 \vee V_2$ and let $I_1 = \{i : C_{2i+1} \in V_1\}$ and $I_2 = \{i : C_{2i+1} \in V_2\}$. If both I_1 and I_2 are finite then there exists m such that $C_{2m+1} \notin V_1$ and $C_{2m+3} \notin V_2$, which is impossible. So either I_1 is infinite or I_2 is infinite. Suppose without loss of generality that I_1 is infinite; then $C_{2i+1} \in V_1$ for all $i \geq 1$. Since C_{2i} is a subalgebra of C_{2i+1}, then $C_{2i} \in V_1$ for all $i \geq 1$. Then $C_j \in V_1$ for all $j \geq 1$. Consequently, $\underset{\sim}{K} \subseteq V_1$. Then $\underset{\sim}{K} = V_1$, and $\underset{\sim}{K}$ is join-irreducible.
A similar argument shows that $\underset{\sim}{P}$ and $\mathcal{L}$ are join-irreducible. □

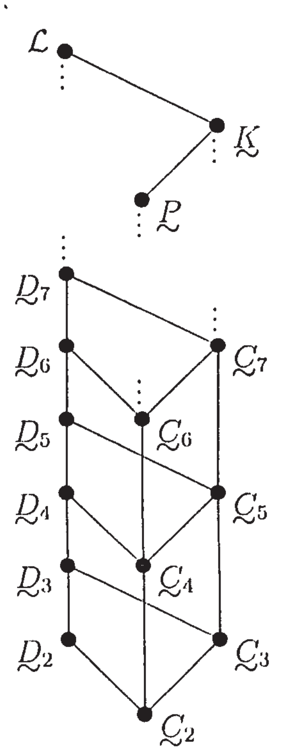

Fig. 1

Observe that from the definitions of $\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathcal{L}$, it follows that they are not completely join-irreducible. Furthermore, they are not finitely generated.

Theorem 2.4 *$\underset{\sim}{K}$, $\underset{\sim}{P}$ and $\mathcal{L}$ are the unique join-irreducible varieties that are not finitely generated.*

Proof. Let V be a join-irreducible not finitely generated variety, $V \subseteq \mathcal{L}$. Consider the following sets:

$$I_1 = \{n : D_n \in V\},\ I_2 = \{n : C_{2n+1} \in V\},\ I_3 = \{n : C_{2n} \in V\},$$

and consider

$$U_{I_1} = V\left(\{D_n : n \in I_1\}\right),\ U_{I_2} = V\left(\{C_{2n+1} : n \in I_2\}\right),\ U_{I_3} = V\left(\{C_{2n} : n \in I_3\}\right)$$

If I_1, I_2 and I_3 are finite, then V is finitely generated, which is a contradiction.
If I_1 is infinite, then $D_n \in V$ for all n, and consequently $C_n \in V$ for all n. Hence, $V = \mathcal{L}$.
Suppose therefore that I_1 is finite, and let m_1 be the greatest element in I_1. If I_2 is infinite, then $C_{2n+1} \in V$ for all n, and $C_{2n} \in V$ for all n. Then $\underset{\sim}{K} = U_{I_2}$, and $V = \underset{\sim}{K} \vee V\,(\bigcup_{i=1}^{m_1} D_i)$. Since V is join-irreducible infinitely generated, then $V = \underset{\sim}{K}$.
If I_2 is finite, let m_2 be the greatest element in I_2. Then necessarily I_3 is infinite, and hence $\underset{\sim}{P} \subseteq V$. Therefore $V = \underset{\sim}{P} \vee V\,(\bigcup_{i=1}^{m_2} C_{2i+1}) \vee V\,(\bigcup_{i=1}^{m_1} D_i)$. Again, since V is join-irreducible infinitely generated, $V = \underset{\sim}{P}$, and the proof is complete. □

The following Theorem will be important in the determination of equational bases for join-irreducible varieties.

Theorem 2.5 *Every variety $V \in \Lambda(\mathcal{L})$ is a join of finitely many varieties in $\mathcal{J}(\Lambda(\mathcal{L}))$.*

Proof. Let $V \in \Lambda(\mathcal{L})$ and let I_1, I_2, I_3 be defined as in the proof of the previous Theorem. If I_1, I_2 and I_3 are all finite, then $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{C}_{2m_2+1} \vee \underset{\sim}{C}_{2m_3}$, where m_1, m_2 and m_3 are the greatest elements in I_1, I_2 and I_3 respectively.
If I_1 is infinite, then $V = \mathcal{L}$.
Suppose that I_1 is finite. If I_2 is finite, then $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{K}$. If I_2 is finite and I_3 is infinite, then $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{C}_{2m_2+1} \vee \underset{\sim}{P}$, as required. □

So we can conclude that if $V \in \Lambda(\mathcal{L})$, then V is of one of the following forms:

(a) $\mathcal{L}$, $\underset{\sim}{K}$, $\underset{\sim}{P}$, $\underset{\sim}{D}_n$, $\underset{\sim}{C}_{2n+1}$, $\underset{\sim}{C}_n$, for V join-irreducible.

(b) $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{C}_{2m_2+1} \vee \underset{\sim}{C}_{2m_3}$, for V finitely generated, join-reducible
and $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{C}_{2m_2+1} \vee \underset{\sim}{P}$, $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{K}$, $V = \underset{\sim}{D}_{m_1} \vee \underset{\sim}{P}$, $V = \underset{\sim}{C}_{m_2+1} \vee \underset{\sim}{P}$,
for V infinitely generated, join-reducible.

3 Equational bases

In this section we will find equational bases for each subvariety of $\mathcal{L}$.

A. Monteiro ([9], p.125) proved that the equation $\gamma_{\underset{\sim}{K}}(x) = \neg x \Rightarrow\sim x = 1$ determines the equational class $\underset{\sim}{K}$.

On the other hand, let $\gamma_{\underset{\sim}{P}}(x)$ be the equation

$$\gamma_{\underset{\sim}{P}}(x) = \neg\sim(\sim x \Rightarrow x) \Rightarrow \neg\neg\sim(x \Rightarrow\sim x) = 1.$$

If $x \in C_{2n}$ and $x >\sim x$, then $x \Rightarrow\sim x = \sim x$, and $\sim x \Rightarrow x = 1$. Thus

$$\gamma_{\underset{\sim}{P}}(x) = \neg\sim 1 \Rightarrow \neg\neg\sim\sim x = 1 \Rightarrow \neg\neg x$$

. Since $x \neq 0$, $\neg\neg x = 1$, and then $\gamma_{\underset{\sim}{P}}(x) = 1$. If $x < \sim x$ we also have $\gamma_{\underset{\sim}{P}}(x) = 1$.
So the equation $\gamma_{\underset{\sim}{P}}(x) = 1$ holds in all algebras C_{2n}.

Now, for the algebras C_{2n+1}, choose the element $c \in C_{2n+1}$ such that $\sim c = c$. Then $\gamma_{\underset{\sim}{P}}(c) = 0$, and therefore the equation $\gamma_{\underset{\sim}{P}}(x) = 1$ fails in the algebras C_{2n+1}.

Then we have the following Theorem:

Theorem 3.1 (A. Monteiro): *The equations $\gamma_{\underset{\sim}{K}}(x) = 1$ and $\gamma_{\underset{\sim}{P}}(x) = 1$ characterize the variety $\underset{\sim}{P}$ within $\mathcal{L}$.*

As we pointed out in Section 1, the variety I_n is generated by $D_n \times D_{n-1}$. Nevertheless, we proved that D_{n-1} is a subalgebra of D_n, thus I_n is the variety generated by D_n, that is, the variety I_n is the variety $\underset{\sim}{D}_n$, and consequently, the Ivo Thomas identity $\gamma_n(x_0, \ldots, x_{n-1}) = 1$ determines the variety $\underset{\sim}{D}_n$, for $n \geq 2$.

The variety that A. Monteiro called $\underset{\sim}{I}_n\underset{\sim}{K}$ is defined as the subvariety of $\mathcal{L}$ characterized by the identities $\gamma_n(x_0, \ldots, x_{n-1}) = 1$ and $\gamma_{\underset{\sim}{K}}(x) = 1$. Monteiro proved ([9], p. 152, Th. 1.1) that $\underset{\sim}{I}_n\underset{\sim}{K} = V(C_n) = \underset{\sim}{C}_n$, for n odd, and $\underset{\sim}{I}_n\underset{\sim}{K} = V(C_n \times C_{n-1})$, for n even. Then we have:

Theorem 3.2 *The equations $\gamma_n(x_0, \ldots, x_{n-1}) = 1$ and $\gamma_{\underset{\sim}{K}}(x) = 1$ determine the variety $\underset{\sim}{C}_n$ for n odd, and the equations $\gamma_n(x_0, \ldots, x_{n-1}) = 1$, $\gamma_{\underset{\sim}{K}}(x) = 1$ and $\gamma_{\underset{\sim}{P}}(x) = 1$ determine the variety $\underset{\sim}{C}_n$ for n even.*

The above results have given us equational bases for all join-irreducible algebras in the lattice $\Lambda(\mathcal{L})$. Our next objective is to find equational bases for the rest of the varieties, that is, the join-reducible varieties.

Let $V \in \Lambda(\mathcal{L})$ and assume that V is finitely generated. Suppose that $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2s+1} \vee \underset{\sim}{C}_{2t}$, where $2t > 2s+1 > r$. Observe that if $2t < r$ then $\underset{\sim}{C}_{2t} \subseteq \underset{\sim}{D}_r$, and if $2t < 2s+1$ then $\underset{\sim}{C}_{2t} \subseteq \underset{\sim}{C}_{2s+1}$. In both cases $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2s+1}$. Finally, if $2s+1 < r$, then $\underset{\sim}{C}_{2s+1} \subseteq \underset{\sim}{D}_r$ and then $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2t}$.

Consider the following identity:

$$\gamma_V(x_0, \ldots, x_{r-1}, y_0, \ldots, y_{2s+1}, z_0, \ldots, z_{2t+1}) =$$

$$\gamma_r(x_0, \ldots, x_{r-1}) \vee \Big(\gamma_{2s+1}(y_0, \ldots, y_{2s}) \wedge \gamma_{\underset{\sim}{K}}(y_{2s+1})\Big) \vee \Big(\gamma_{2t}(z_0, \ldots, z_{2t-1}) \wedge \gamma_{\underset{\sim}{K}}(z_{2t}) \wedge \gamma_{\underset{\sim}{P}}(z_{2t+1})\Big)$$

If $A \in \mathrm{Si}_{\mathrm{fin}}(V)$, then $A \in \underset{\sim}{D}_r$ or $A \in \underset{\sim}{C}_{2s+1}$ or $A \in \underset{\sim}{C}_{2t}$.
Hence, for any $x_0, \ldots, x_{r-1}, y_0, \ldots, y_{2s+1}, z_0, \ldots, z_{2t+1} \in A$, one of the identities $\gamma_r = 1$ or $\gamma_{2s+1} \wedge \gamma_{\underset{\sim}{K}} = 1$ or $\gamma_{2t} \wedge \gamma_{\underset{\sim}{K}} \wedge \gamma_{\underset{\sim}{P}} = 1$ holds. So $\gamma_V = 1$ holds in A, and consequently, $\gamma_V = 1$ holds in V.

Let $A \in \mathrm{Si}_{\mathrm{fin}}(\mathcal{L})$ such that $A \notin V$. Then we have the following cases:

I. $A = C_l$

1. l even, $l > 2t$ (in particular, $l > 2s+1$, $l > r$).
Then A does not satisfy the identities $\gamma_r = 1$, $\gamma_{2s+1} = 1$ and $\gamma_{2t} = 1$. In other words, there exist elements $x_0, \ldots, x_{r-1}, y_0, \ldots, y_{2s}, z_0, \ldots, z_{2t-1} \in A$ such that $\gamma_r(x_0, \ldots, x_{r-1}) = a \neq 1$, $\gamma_{2s+1}(y_0, \ldots, y_{2s}) = b \neq 1$ and $\gamma_{2t}(z_0, \ldots, z_{2t-1}) = c \neq 1$. Then $\gamma_V(x_0, \ldots, x_{r-1}, y_0, \ldots, y_{2s}, y_{2s+1}, z_0, \ldots, z_{2t-1}, z_{2t}, z_{2t+1}) \leq a \vee b \vee c \neq 1$, since $A = C_l$ is a chain, $a \neq 1$, $b \neq 1$, $c \neq 1$, a, b, $c \in A$.
2. l odd, $l > 2s+1$ (in particular, $l > r$).
Then A does not satisfy the identities $\gamma_{\underset{\sim}{P}} = 1$, $\gamma_{2s+1} = 1$ and $\gamma_r = 1$.
A similar argument shows that A does not satisfy γ_V.

II. $A = D_l = C_l \times C_l$, $l > r$.
Then the identity $\gamma_r = 1$ fails in A, that is, there exist $x_0, \ldots, x_{r-1} \in A$ such that $\gamma_r(x_0, \ldots, x_{r-1}) = a \neq 1$, $a \in C_l \times C_l$. Put $a = (b_1, b_2)$, b_1, $b_2 \in C_l$, $(b_1, b_2) \neq (1,1)$. Suppose $b_1 \neq 1$. Then choose $y_{2s+1} = (0,1) = z_{2t}$. Then $\gamma_{\underset{\sim}{K}}(y_{2s+1}) = \gamma_{\underset{\sim}{K}}(z_{2t}) = \neg(0,1) \Rightarrow\sim (0,1) = (0 \Rightarrow 0, 1 \Rightarrow 0) \Rightarrow (1-1, 1-0) = (1,0) \Rightarrow (0,1) = (0,1)$. Thus $\gamma_V \leq (b_1, b_2) \vee (0,1) = (b_1, 1) \neq (1,1)$.
If $b_2 \neq 1$, we choose $y_{2s+1} = z_{2t} = (1,0)$.

So we have proved that the identity $\gamma_V = 1$ determines the variety V.

In a quite similar way, it can be proved that:

1. If $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2s+1}$, $2s+1 > r$, then $\gamma_V = \gamma_r \vee (\gamma_{2s+1} \wedge \gamma_{\underset{\sim}{K}}) = 1$.
2. If $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2t}$, $2t > r$, then $\gamma_V = \gamma_r \vee (\gamma_{2t} \wedge \gamma_{\underset{\sim}{K}} \wedge \gamma_{\underset{\sim}{P}}) = 1$.
3. If $V = \underset{\sim}{C}_{2s+1} \vee \underset{\sim}{C}_{2t}$, $2t > 2s+1$, then $\gamma_V = (\gamma_{2s+1} \wedge \gamma_{\underset{\sim}{K}}) \vee (\gamma_{2t} \wedge \gamma_{\underset{\sim}{K}} \wedge \gamma_{\underset{\sim}{P}}) = 1$.

Therefore, we have given equational bases for every finitely generated subvariety of $\mathcal{L}$.

Now, if V is an infinitely generated subvariety of $\mathcal{L}$, we have seen that $\underset{\sim}{P} \subseteq V$ or $\underset{\sim}{K} \subseteq V$ or $V = \mathcal{L}$.
Suppose that $V = \underset{\sim}{D}_r \vee \underset{\sim}{C}_{2t+1} \vee \underset{\sim}{P}$. We can assume that $2t+1 > r$. Then the identity $\gamma_V = \gamma_r \vee (\gamma_{2t+1} \wedge \gamma_{\underset{\sim}{K}}) \vee (\gamma_{\underset{\sim}{P}} \wedge \gamma_{\underset{\sim}{K}}) = 1$ determines the variety V.
For $V = \underset{\sim}{D}_r \vee \underset{\sim}{P}$, an equational basis for V is given by $\gamma_V = \gamma_r \vee (\gamma_{\underset{\sim}{P}} \wedge \gamma_{\underset{\sim}{K}}) = 1$.
For $V = \underset{\sim}{C}_{2t+1} \vee \underset{\sim}{P}$, $\gamma_V = (\gamma_{2t+1} \wedge \gamma_{\underset{\sim}{K}}) \vee (\gamma_{\underset{\sim}{P}} \wedge \gamma_{\underset{\sim}{K}}) = 1$.
For $V = \underset{\sim}{D}_r \vee \underset{\sim}{K}$, $\gamma_V = \gamma_r \vee \gamma_{\underset{\sim}{K}} = 1$.

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