RUIN PROBABILITIES AND OPTIMAL STOPPING FOR A DIFFUSION WITH JUMPS

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ABSTRACT. In this paper we give the ruin probabilities for a diffusion with jumps. As a related problem, we are able to give closed form solution of some optimal stopping problems for the same processes.

Keywords and Phrases: Diffusion with Jumps, Ruin Probabilities, Barrier Problems, Optimal stopping, Derivetive Pricing.

1. Introduction.

Let be given on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a Wiener process $W = (W_t)_{t\geq 0}$, a Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$, and a sequence of independent nonnegative random variables $Y = (Y_k)_{k\geq 1}$, with identical distribution F. We will denote $F \sim \exp(\alpha)$ when F is exponential with parameter $\alpha > 0$. Assume that the processes W, N and Y are independent, and that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the minimal filtration satisfying the usual conditions (see Jacod and Shiryaev (1987), page 2) such that the process $X = (X_t)_{t\geq 0}$ given by

$$X_t = x + \sigma W_t + \sum_{k=1}^{N_t} Y_k - at, \qquad t \geq 0, \tag{1.1}$$

is adapted to $\mathbb{F}$. Here x, σ, and a are real constants with σ and a positive.

τ is a stopping time (or stopping rule) relative to $\mathbb{F}$, if

$$\tau: \Omega \to [0, +\infty] \quad \text{and} \quad \{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all} \quad t \geq 0.$$

Denote by $\mathcal{M}$ the class of all stopping times.

The first question faced in this paper is the computation of the following ruin probabilities (see Feller (1966)):

$$R^+(x) = P(\exists t: X_t \leq 0), \tag{1.2}$$

and

$$R^-(x) = P(\exists t: X_t \geq 0). \tag{1.3}$$

The exact solution to (1.2) is well known even for general process with stationary independent increments, see for instance Prabhu (1980), but the solution to (1.3) seems to be new (see anyway Skorohod (1991)).

The second problem is the following : given a Borel function $g : \mathbb{R} \to \mathbb{R}$ find a real function $s : \mathbb{R} \to \mathbb{R}$ and a stopping rule τ^*, such that

$$s(x) = \sup_{\tau \in \mathcal{M}} E(g(X_\tau)) = E(g(X_{\tau^*})). \tag{1.4}$$

Here s is called the cost function, and τ^* (the stopping time that realizes the supremum) the optimal stopping rule.

In the present paper we give the closed form solution to the problem (1.4) when the function g is given either by

$$g(x) = (x - K)^+,$$

or

$$g(x) = (K - x)^+,$$

with K a real constant.

The results on optimal stopping presented are related to those in Mordecki (1997), where different functions g (used in stochastic finance) were considered.

For similar results see Mc Kean (1965), Zhang (1995), Mordecki (1996), and Shiryaev (1978) for general reference on the subject.

The paper is organized as follows. In section 2 we present the main results. The proofs, given in section 4 are based on some preliminary results, given in section 3.

2. Main Results.

2.1. Ruin Probabilities.

Theorem 1. *Let X be given by (1.1) with $\mu = \int_0^{+\infty} y dF(y)$ and $a < \lambda\mu$. Then the ruin probability (1.2) is given by*

$$R^+(x) = \begin{cases} e^{\eta x} & \text{if } x \geq 0, \\ 1 & \text{if } x < 0, \end{cases} \tag{2.1}$$

for η the negative root of the equation

$$\frac{\sigma^2}{2}\eta^2 - a\eta + \lambda \int_0^{+\infty} (e^{\eta y} - 1) dF(y) = 0. \tag{2.2}$$

Theorem 2. *Let X be given by (1.1) with $F \sim \exp(\alpha)$, and $\lambda < a\alpha$. Then the ruin probability (1.3) is given by*

$$R^-(x) = \begin{cases} Ae^{-p_1 x} + Be^{-p_2 x}, & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \tag{2.3}$$

with p_1 *and* p_2 *the roots of*

$$Q(p) = \frac{\sigma^2}{2}p^2 + (\frac{\alpha\sigma^2}{2} + a)p + \alpha a - \lambda = 0, \tag{2.4}$$

and A, B *given by*

$$A = \frac{p_2(p_1+\alpha)}{\alpha(p_2-p_1)}, \qquad B = \frac{p_1(p_2+\alpha)}{\alpha(p_1-p_2)}.$$

Remark: From Theorem 2 we obtain the distribution of the supremum of the process X, when $F \sim \exp(\alpha)$. Denote by $S = \sup_{t\geq 0} X_t$. Let $y > x$, then

$$P(S \geq y) = P(\exists t \colon x + \sigma W_t + \sum_{k=1}^{N_t} Y_k - at \geq y) =$$

$$P(\exists t \colon y - x + \sigma W_t + \sum_{k=1}^{N_t} Y_k - at \geq 0) = R^-(x-y).$$

In the proof of Theorem 3, we use the fact that

$$E(S) = -A\frac{e^{-p_1 x}}{p_1} - B\frac{e^{-p_2 x}}{p_2}.$$

For further reference (Mordecki (1977)), it is easy to compute that

$$E(e^S) = -A\frac{e^{-p_1 x}}{p_1+1} - B\frac{e^{-p_2 x}}{p_2+1} < +\infty.$$

(Observe that the result is finite because $p_i + 1 < -1$ for $i = 1, 2$).

2.2. Optimal Stopping.

Theorem 3. *Let* X *be given by (1.1), and* $g(x) = (x-K)^+$. *Assume that* $F \sim \exp(\alpha)$, *and* $\lambda < a\alpha$. *Denote*

$$x_0 = K + \frac{\lambda + \frac{\alpha^2\sigma^2}{2}}{\alpha(a\alpha - \lambda)}. \tag{2.5}$$

Then, the solution to the optimal stopping problem (1.4) is

$$\tau^* = \inf\{t \geq 0 : X_t \geq x_0\},$$

$$s(x) = \begin{cases} Ae^{p_1(x_0-x)} + Be^{p_2(x_0-x)}, & \text{if } x \leq x_0, \\ x - K, & \text{if } x > x_0. \end{cases} \tag{2.6}$$

where $p_2 < p_1 < 0$ *are the roots of the equation (2.4), and the coefficients* A *and* B *are*

$$A = \frac{p_2(x_0-K)+1}{p_2-p_1}, \qquad B = \frac{p_1(x_0-K)+1}{p_1-p_2}.$$

Theorem 4. *Let X be given by (1.1) and $g(x) = (K - x)^+$. Assume that F is arbitrary. Denote by $m = \int_0^{+\infty} y dF(y)$, and suppose*

$$a < \lambda m. \tag{2.8}$$

Let $\eta < 0$ be the unique root of the equation (2.2). Denote

$$x_0 = K + \frac{1}{\eta}.$$

Then, the solution to the optimal stopping problem (1.4) is

$$\tau^* = \inf\{t \geq 0 : X_t \leq x_0\},$$

$$s(x) = \begin{cases} K - x, & \text{if } x \leq x_0, \\ (K - x_0)\exp\{\eta(x - x_0)\}, & \text{if } x > x_0. \end{cases} \tag{2.9}$$

3. Some preliminary results.

3.1. In order to prove Theorems 1 and 2 we have to apply Itô's formula to the process X defined in (1.1) and the functions R, defined in (2.1) and (2.3) respectively (for simplicity, we drop the superindex $+$ and $-$ when not neccesary). As the function R is not $\mathbf{C}^2(\mathbb{R})$ we will apply Meyer-Itô formula (Theorem IV.51 of Protter (1992)). The second derivative of R in the sense of distributions when restricted to compacts is

$$\mu(da) = R''(a)da + R_0\delta_0(da)$$

with R'' the second derivative of R if $a \neq 0$, $R_0 = R'(0^+) - R'(0^-)$, is the difference between the lateral limits of the first derivative, and δ_0 is the point mass at 0.

So, Meyer-Itô formula gives

$$\begin{aligned} R(X_t) - R(x) = & \int_0^t R'(X_{s-})dX_s + \\ & + \sum_{0 \leq s \leq t} (R(X_s) - R(X_{s-}) - R'(X_{s-})\Delta X_s) \\ & \frac{1}{2}\int_{\mathbb{R}} l^X(a,t)\mu(da) = \\ & = \int_0^t R'(X_{s-})dX_s + \\ & + \sum_{0 \leq s \leq t} (R(X_s) - R(X_{s-}) - R'(X_{s-})\Delta X_s) \\ & \frac{1}{2}\int_0^t R''(X_{s-})d\langle X, X\rangle_s + \frac{1}{2}R_0 l^X(0,t). \end{aligned}$$

where $l^X(a,t)$ is the local time of the process X at level a and time t. If

$$\tau_0 = \inf\{t \geq 0 \colon X_t \leq 0\},$$

we have $l^X(0,\tau_0) = 0$ because $X_t > 0$ on $[0,\tau_0)$. Then

$$R(X_{\tau_0 \wedge t}) - R(x) = \int_0^{\tau_0 \wedge t} R'(X_{s-})dX_s + \frac{1}{2}\int_0^{\tau_0 \wedge t} R''(X_{s-})d\langle X, X\rangle_s$$
$$+ \sum_{0 \leq s \leq \tau_0 \wedge t} \left(R(X_s) - R(X_{s-}) - R'(X_{s-})\Delta X_s\right).$$

Furthermore, denoting $X^c = (X_t^c)_{t\geq 0}$, $X^d = (X_t^d)_{t \geq 0}$, with

$$X_t^c = x + \sigma W_t + at, \quad \text{and} \quad X_t^d = \sum_{i=1}^{N_t} Z_i,$$

we have:

$$\int_0^{\tau_0 \wedge t} R'(X_{s-})dX_s + \sum_{0 \leq s \leq \tau_0 \wedge t} \left(R(X_s) - R(X_{s-}) - R'(X_{s-})\Delta X_s\right) =$$

$$\int_0^{\tau_0 \wedge t} R'(X_{s-})dX_s^c +$$
$$\int_0^{\tau_0 \wedge t} \int_{\mathbb{R}} [R(X_{s-} + x) - R(X_{s-})] * (\mu(\omega, dx, ds) - \nu(dx, ds)) +$$

$$\int_0^{\tau_0 \wedge t} \int_{\mathbb{R}} [R(X_{s-} + x) - R(X_{s-})] * \nu(dx, ds),$$

where $\mu = \mu(\omega, dx, ds)$ is the jump measure corresponding to X^d, and

$$\nu = \nu(dx, dt) = \lambda dt F(dx)$$

its compensator. Resuming, Itô's formula in our case reads

$$R(X_{\tau_0 \wedge t}) - R(x) = \int_0^{\tau_0 \wedge t} (L^X R)(X_{s-})ds + M(R)_{\tau_0 \wedge t} \tag{3.1}$$

with

$$(L^X R)(x) = \frac{1}{2}\sigma^2 R''(x) + aR'(x) + \lambda \int_{\mathbb{R}} (R(x+y) - R(x))dF(x) \tag{3.2}$$

the infinitesimal generator of the process X, and the local martingale $M(R) = (M(R)_t)_{t \geq 0}$ given by

$$M(R)_t = \sigma \int_0^t R'(X_{s-})dW_s + \int_0^t \int_{\mathbb{R}} (R(X_{s-} + x) - R(X_{s-})) * (\mu - \nu). \tag{3.3}$$

3.2. In order to apply Itô's formula to the process X defined in (1.1) and the functions s defined by (2.6) and (2.9) it is enough to observe that, although s is not $\mathbf{C}^2(\mathbb{R})$, $s''(x)$ is continuous for $x \neq x_0$, and has finite lateral limits. This gives, denoting $\mu(da)$ the signed measure (when restricted to compacts) which is the second derivative of s in the generalized sense,

$$\mu(da) = s''(a)da.$$

So Meyer-Itô formula applies, and a minor modifications in Corollary 1 to the same formula, necessary because s is only bounded on compacts, gives

$$s(X_t) - s(x) = \int_0^t s'(X_{s-})dX_s + \frac{1}{2}\int_0^t s''(X_{s-})d\langle X, X\rangle_s + \sum_{0\leq s\leq t} (s(X_s) - s(X_{s-}) - s'(X_{s-})\Delta X_s).$$

In this case, following the notation introduced in **3.1** we have

$$\int_0^t s'(X_{s-})dX_s + \sum_{0\leq s\leq t} (s(X_s) - s(X_{s-}) - s'(X_{s-})\Delta X_s) =$$
$$\int_0^t s'(X_{s-})dX_s^c +$$
$$\int_0^t \int_{\mathbb{R}} [s(X_{s-} + x) - s(X_{s-})] * (\mu(\omega, dx, ds) - \nu(dx, ds)) +$$
$$\int_0^t \int_{\mathbb{R}} [s(X_{s-} + x) - s(X_{s-})] * \nu(dx, ds).$$

So, Itô's formula reads

$$s(X_t) - s(x) = \int_0^t (Ls)(X_{s-})ds + M(s)_t. \tag{3.4}$$

with L^X and $M(\cdot)$ defined in (3.2) and (3.3).

In order to prove Theorems 3 and 4, we need the following Lemma (see Mordecki (1997)), that we include for the sake of completeness.

Lemma 3.1. *Let $X = (X_t)_{t\geq 0}$ be given by (1.1). Let s and g be real Borel functions, such that s is convex with s'' continuous for $x \neq x_0$, for some $x_0 \in \mathbb{R}$, and has finite lateral limits. C^*, the continuation region is an open halfline of the form $(-\infty, x_0)$ or $(x_0, +\infty)$. Let*

$$\tau^* = \inf\{t \geq 0: X_t \notin C^*\}.$$

Assume that the following five conditions hold:

(1) $(Ls)(x) = 0 \qquad \forall x \in C^*$.

(2) $(Ls)(x) \leq 0 \qquad \forall x \neq x_0$.

(3) $0 \leq g(x) \leq s(x) \qquad \forall x \in \mathbb{R}$.

(4) $s(X_{\tau^* \wedge T \wedge t}) \leq Z$ *P-a.s for all* $T \in \mathcal{M}$ *and for all* $t \in \mathbb{R}^+$, *with* Z *an integrable random variable, that is* $E|Z| < +\infty$.

(5) $s(X_{\tau^*}) = g(X_{\tau^*})$ *P-a.s.*

Then, under this assumptions, the pair (τ^*, s) *is the solution for the optimal stopping problem (1.4) for the function* g *and the process* X, *that is:*

$$s(x) = \sup_{\tau \in \mathcal{M}} E(g(X_\tau)) = E(g(X_{\tau^*})).$$

Proof. By (3.4)

$$s(X_t) - s(x) = \int_0^t (Ls)(X_{s-})ds + M(s)_t.$$

We have to prove assertions (a) and (b):

(a) $s(x) = Eg(X_{\tau^*})$,

(b) $s(x) \geq Eg(X_\sigma) \;\; \forall \sigma \in \mathcal{M}$.

By conditions (5) and (3) in our hypothesis this is equivalent to proving

(a') $s(x) = Es(X_{\tau^*})$,

(b') $s(x) \geq Es(X_\sigma) \;\; \forall \sigma \in \mathcal{M}$.

Taking into account that $A_t = -\int_0^t (Ls)(X_{s-})ds$ is increasing (by condition (2)) and $A_{\tau^*} = 0$ (by condition (1)) we have to verify

(a") $E(M_{\tau^*}) = 0$,

(b") $E(M_\sigma) \leq 0 \;\; \forall \sigma \in \mathcal{M}$.

As $M(s) = (M(s)_t)_{t \geq 0}$ is a local martingale with $M(s)_0 = 0$, for a localizing sequence (τ_n) we have

$$E(M(s)_{t \wedge \tau^* \wedge \tau_n}) = E(M(s)_0) = 0.$$

As

$$-s(x) \leq M(s)_{t \wedge \tau^* \wedge \tau_n} \leq s(X_{t \wedge \tau^* \wedge \tau_n}) \leq Z$$

(a") follows by dominated convergence.

As $M(s)_t \geq -s(x)$, by Fatou's Lemma the local martingale $M(s)$ is in fact a supermartingale with $EM(s)_0 = 0$ and (b") follows.

4. Proof of the theorems.

Proof of Theorem 1. Taking into account (3.2), for $x > 0$

$$L^X e^{\eta x} = e^{\eta x} \big[\frac{1}{2}(\eta\sigma)^2 - a\eta + \lambda \int_0^{+\infty} (e^{\eta y} - 1)dF(y)\big] = 0.$$

When $0 \leq s \leq \tau_0 \wedge t$ we have $X_{s-} > 0$, and in consequence $L^X(X_{s-}) = 0$. Also $l^X(0, \tau_0) = 0$, so (3.1) reads

$$R(X_{\tau_0 \wedge t}) - R(x) = M(R)_{\tau_0 \wedge t}$$

As R is a bounded function, taking expected values and limit as t goes to infinity

$$R(x) = E(R(X_{\tau_0})) = E(R(X_{\tau_0})\mathbb{I}_{\{\tau_0<\infty\}}) + E(R(X_{\tau_0})\mathbb{I}_{\{\tau_0=\infty\}}) = P(\tau_0 < \infty).$$

Proof of Theorem 2. As $A + B = 1$, defining

$$R_i(x) = \mathbb{I}_{\{x \geq 0\}} e^{p_i x} + \mathbb{I}_{\{x<0\}}, \quad i = 1, 2.$$

we have $R = AR_1 + BR_2$. For $x < 0$:

$$L^X R_i(x) = p_i e^{p_i x} Q(p_i) + \alpha\lambda \frac{p_i}{p_i + \lambda} = \alpha\lambda \frac{p_i}{p_i + \lambda},$$

so

$$L^X R(x) = \alpha\lambda \frac{Ap_1}{p_1 + \lambda} + \alpha\lambda \frac{Bp_2}{p_2 + \lambda} = 0$$

by (2.5), and the proof goes as in Theorem 1.

In view of Lemma 3.1, the proofs of Theorems 3 and 4 reduces to the verification of conditions (1) to (5) in each case.

Proof of Theorem 3.

(1) For $x < x_0$ and $s(x)$ as in (2.6) we have

$$\begin{aligned}
(L^X s)(x) &= \frac{1}{2}\sigma^2 s''(x) - as'(x) + \lambda \int_0^{+\infty} [s(x + y) - s(x)]\alpha e^{-\alpha y} dy = \\
& Ae^{p_1(x_0-x)} \big[\frac{1}{2}\sigma^2 p_1^2 + ap_1 - \lambda \frac{p_1}{p_1 + \alpha}\big] + \\
& Be^{p_2(x_0-x)} \big[\frac{1}{2}\sigma^2 p_2^2 + ap_2 - \lambda \frac{p_2}{p_2 + \alpha}\big] - \\
& \lambda e^{-\alpha(x_0-x)} \big[x_0 - K + \frac{1}{\alpha} - \alpha \frac{A}{p_1 + \alpha} - \alpha \frac{B}{p_2 + \alpha}\big] = \\
& A \frac{p_1}{p_1 + \alpha} e^{p_1(x_0-x)} Q(p_1) + B \frac{p_2}{p_2 + \alpha} e^{p_2(x_0-x)} Q(p_2) - \\
& \lambda e^{-\alpha(x_0-x)} \big[x_0 - K + \frac{1}{\alpha} - \alpha \frac{A}{p_1 + \alpha} - \alpha \frac{B}{p_2 + \alpha}\big].
\end{aligned}$$

We know $Q(p_1) = Q(p_2) = 0$.

$$x_0 - K + \frac{1}{\alpha} - \alpha \frac{A}{p_1 + \alpha} - \alpha \frac{B}{p_2 + \alpha} = 0$$

is verified taking into account that

$$(p_1+\alpha)(p_2+\alpha) = -\frac{2\lambda}{\sigma^2},$$
$$p_1p_2 = \frac{2(a\alpha-\lambda)}{\sigma^2},$$
$$p_1A+p_2B = -1,$$

and (2.5).

(2) We have to see that for $x > x_0$, $L^X s(x) \le 0$:

$$(L^X s)(x) = -a + \lambda \int_0^{+\infty} y\,dF(y) = -a + \frac{\lambda}{\alpha} < 0$$

by hipothesis.

(3) First, we verify that

$$A > 0 \quad \text{and} \quad B > 0. \tag{4.1}$$

We know $A+B = x_0 - K > 0$. On the other side,

$$\begin{aligned} AB &= \frac{-1}{(p_1-p_2)^2}\left[p_2(x_0-K)+1\right]\left[p_1(x_0-K)+1\right] \\ &= \frac{-2}{\sigma^2(p_1-p_2)^2(x_0-K)^2}Q\left(\frac{-1}{x_0-K}\right). \end{aligned}$$

Now, $Q(\frac{-1}{x_0-K}) < 0$ by (2.5) and (4.1) is proved.

(4) Take $Z = s(X_{\tau^*}) + x_0$. Then, $s(X_{\tau^*\wedge T\wedge t}) \le Z$ follows from the fact that on the set $\{\tau^* < +\infty\}$ we have $X_{\tau^*\wedge T\wedge t} \le X_{\tau^*}$ and the function s is increasing. On the set $\{\tau^* = +\infty\}$ we have $X_{\tau^*\wedge T\wedge t} \le x_0$.

Observe now, that

$$s(X_{\tau^*}) \le (X_{\tau^*} - K)^+ \le \sup_{0\le t\le +\infty} X_t$$

and by Remark to Theorem 2 we have $E(\{\sup_{0\le t\le+\infty} X_t\}) < \infty$.

(5) We have

$$\lim_{x\to-\infty} s(x) = \lim_{x\to-\infty} g(x) = 0$$

so $s(X_{\tau^*}) = g(X_{\tau^*}) = 0$ on $\{\tau^* = +\infty\}$ because $\lim_{t\to+\infty} X_t = -\infty \quad P-a.s.$, and $s(x) = g(x)$ if $x \ge x_0$, so $s(X_{\tau^*}) = g(X_{\tau^*})$ on $\{\tau^* < +\infty\}$.

Proof of Theorem 4.

(1) If $s(x) = (K-x_0)e^{\eta(x-x_0)}$ for $x > x_0$

$$(L^X s)(x) = (K-x_0)e^{\eta(x-x_0)}\left[\frac{(\sigma\eta)^2}{2} - a\eta + \lambda\int_0^{+\infty}(e^{\eta y}-1)dF(y)\right] = 0$$

by (2.2).

(2) In this case, for $x < x_0$

$$(L^X s)(x) = -a + \lambda \int_0^{+\infty} y dF(y) = -a + \lambda m < 0$$

by (2.8).

(3) For $x > K$, $g(x) = 0$. When $x_0 < x < K$, we have $s'' > 0$, with $g'' > 0$, $g(x_0) = s(x_0)$ and $g'(x_0) = s'(x_0)$.

(4) is inmediate, because in this case we have

$$0 \leq s(X_t) \leq x_0.$$

(5) In this case we have $X_{\tau^*} = x_0$ on the set $\{\tau^* < +\infty\}$, and in consequence $g(X_{\tau^*}) = s(X_{\tau^*}) = s(x_0)$ on $\{\tau^* < +\infty\}$. As $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} s(x) = 0$ the result follows.

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