FROM LARGE ECONOMIES TO UNIFORM SEMIGROUP-VALUED MEASURES[1]

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PREMISE. The title is rather binding. Indeed, it evidently promises a connection between two things that, a priori, appear quite different and distant. We shall move along the following lines. We would like to show:

how *perfect competition* leads to a finitely additive measure space of agents;

how the need to extend, to the latter setting, exisiting results on countably additive *set correspondences*, leads to use semigroup-valued measures.

1. Introduction

One of the most representative examples in modern theory of general economic equilibrium is the model of an economy with a finite number of commodities and a finite number of households and firms in the formulation [AD] which is due to the two Nobel Prize winners K. J. Arrow and G. Debreu.

Since our purpose here is to show how economic theory influenced some developments of measure theory, we may remain confined to the simpler case of *exchange economies*. This means that the only economic activity we take into consideration is the exchange of goods among individuals. In other words, economic agents (each starting with a certain initial endowment), for better satisfying personal necessities, exchange commodities untill an equilibrium situation is produced and no more exchanges can take place. We better fix

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notation.

Let us say that Ω is the set of all economic agents and let us assume that there are ℓ commodities. In this way, a typical commodity bundle is a point in the positive cone of $\mathbf{R}^\ell$. The starting situation, namely the distribution of initial endowments is represented by a given function $e(\omega)$ with $\omega \in \Omega$ and values in $\mathbf{R}^\ell_+$. Evidently, the questions arise of:

- characterizing situations $f : \Omega \to \mathbf{R}^\ell_+$ which are equilibria
- proving the existence of equilibria.

Prior to this, we need to model the reason to exchange. This is done by introducing, for each agent, his own *preference relation*[2] $\succeq_\omega$ on the consumption set $\mathbf{R}^\ell_+$. So, it is clear: the reason for any agent ω to exchange is the hope to move from $e(\omega)$ to a point in the set $\{x \in \mathbf{R}^\ell_+ : x \succ_\omega e(\omega)\}$, which would mean an improvement.

Two approaches to equlibrium concept have been traditionally investigated. One is the competitive approach, the other is the cooperative one. They go, respectively, back to L. Walras and to F. Y. Edgeworth.

Roughly speaking, the competitive notion of equilibrium is based on the fact that each individual takes prices as given and acts ignoring other agents. The individual evaluate his own initial endowment at the given price system, then chooses the best bundle available which costs no more than the initial income. Here is the formal definition:

Definition of Walras Equilibrium

A Walras Equilibrium is a pair (f, p) where the situation f and the price system $p \in \mathbf{R}^\ell_+$ are such that

- *total demand equals total supply, namely*

$$\sum_{\omega\in\Omega} f(\omega) = \sum_{\omega\in\Omega} e(\omega)$$

[2]Let me be vague, now, about properties of $\succeq$.

- *and, for any agent* ω, *we have* $p \cdot f(\omega) \leq p \cdot e(\omega)$ *and*

$$x \in \mathbf{R}^{\ell}_{+} \;\; \& \;\; p \cdot x \leq p \cdot e(\omega) \Rightarrow f(\omega) \succeq_{\omega} x.$$

On the other hand, in the Edgeworth's idea, prices do not appear and what matters it is the possibility of a cooperative behaviour of agents. An equilibrium will be a situation that presents no incentive for agents to form coalitions and bargain for a redistribution of resources. More formally, if we set $f(A) = \sum_{\omega \in A} f(\omega)$ for any situation f and any nonempty coalition $A \subseteq \Omega$, then

Definition of Core

A core allocation is any situation f *such that*

- *total demand equals total supply*
- *no coalitions* A *exist for which we can find another situation* g *with* $g(A) = e(A)$ *and* $g(\omega) \succ_{\omega} f(\omega) \; \forall \omega \in A$.

Given the two above concepts of equilibrium, Fundamental Theorems, see [HK], state (under appropriate assumptions) that **Walras equilibria are Core allocations and** that **Walras equilibria do exist.** Moreover, **Core allocations do not necessarily admit prices making them Walras equilibria** (usually is the contrary). This leads us to the basic question:
under which circumstances does cooperative barter and competition through decentralized markets lead essentially to the same result?
Going back to the idea of competitive equilibria we can easily point out that the price-taking behaviour only makes sense when individuals view themselves as an insignificant part of a large market. How to capture this idea? How to model *perfect competition*? The approach, due to Aumann [A] (see also [V]), is to consider an economy in which there are arbitrarily many players none of whom individually makes any noticeable contribution. Formally the assumption will be that Ω will carry nonatomic countably additive measures or, straightly, that it will be identified with the unit interval of the real line.

Although the above assumption might appear artificial, we have to think of this limit economy in the same spirit of physicists, when they treat fluids as a continuum even though they know there are finitely many particles in a given volume.
It is in this contest of perfect competition that Aumann first proved that Core and Walras allocations are the same[3].

2. Role of Measure Theory in Perfect Competition

The fact that the idea of perfect competition was captured by means of modelling the space of economic agents as a nonatomic countably additive measure space, stimulated several mathematical developments. They mainly dealt with the studying of *additive set correspondences* that originates from the economic theory. Already the paper [V] points out two of the principal directions of investigation: richness in selections and geometric structure of the range[4]. A third main direction deals with integration of correspondences and integral representation of additive correspondences.
My personal interest is due to two papers by Armstrong and Richter [AR, AR1] that focused attention on the role that finite additivity should have in perfect competition. The step from countable to finite additivity is not only mathematically important (as it is anyway), it is basic also for its economic relevance. Indeed, as we said in the introduction, the reason to deal with a nonatomic measure space of agents is to depict the limit behaviour of finite economies when their cardinality goes to infinity. Since there are no nonzero nonatomic countably additive measures on the measurable space $(\mathbf{N}, 2^{\mathbf{N}})$, it follows that the countable additivity hypothesis automatically excludes the

[3] We observe that a better understanding of the Core-Walras equivalence theorem requires mentioning the fact that it is possible to formalize that if the cardinality of Ω tends to infinity, then the set of all Core allocations shrinks to the set of all Walras equilibria.

[4] In [V] a Lyapunov-type theorem is proved for the net trade correspondence of an exchange economy, assuming that such correspondence is countably additive, has no atoms and is rich in selections.

possibility of countably many agents. On the other hand it seems clear, and it is usually accepted, that a realistic model for the limit space of economic agents should exactly include the case of countably many of them. The difficulty that comes out is overcome by modelling perfect competition with finitely additive nonatomic measure spaces of players. Of course, the consequence is that it is then necessary to investigate, in the finitely additive setting, the results that the literature on large economies presents in the countably additive case. Partly this has been done in [AB, B, B1] relatively to some of the topics that were crucial from the economic point of view, namely selections and structure of range of set correspondence. Let us go into some details.

3. $\mathbf{R}^\ell$-valued finitely additive set correspondences

Through some theorems for semigroup-valued measures (see the appendix), we can use the classical Stone space argument for deriving, very quickly and easily, the following

Theorem [B, Theorem 3]. *For a closed-valued nonatomic f.a. set correspondence Φ we have:*

a) convexity of values,

b) relative convexity of the range,

c) richness of f.a. selections.

For a f.a. set correspondence we mean a mapping Φ from[5] $\mathcal{F}$ to the family of nonempty subsets of $\mathbf{R}^\ell$ such that $\Phi(\emptyset) = \{0\}$ and

$$\Phi(E \cup F) = \Phi(E) + \Phi(F) = \{x + y : x \in \Phi(E),\ y \in \Phi(F)\}$$

whenever E and F are disjoint.

The range $R(\Phi)$ of Φ is the union of all values of Φ. C_r, $r > 0$, is the ball $\{x \in \mathbf{R}^\ell : |x| \leq r\}$. The meaning of *richness in selection* is clearified in the

[5] $\mathcal{F}$ is a given Boolean algebra of substes of Ω.

next

Definition. *Let S be the set of all contents (= f.a. set functions) that are selections of Φ. We say that Φ is rich in selections if S is nonempty and, moreover,*

$$\Phi(A) = \{\mu(A) : \mu \in S\}$$

for all $A \in \mathcal{F}$.

The definition of nonatomicity for a set correspondence Φ is the following : for any $r > 0$ there exists in $\mathcal{F}$ a partition $\{F_1, \ldots, F_p\}$ of Ω such that

$$E \subseteq F_i, \;\; E \in \mathcal{F} \Rightarrow \Phi(E) \subseteq C_r.$$

It is quite obvious that *a finitely additive nonatomic set correspondence is bounded.*

It appears that a f.a., nonatomic, closed-valued set correspondence Φ is compact valued. This circumstance gives us the possibility to treat such a Φ as an ordinary (semigroup-valued) content (=finitely additive measure, see the appendix). Of course the idea of treating special multifunctions as ordinary functions is standard. Even multimeasures have been treated as vector measures, by means of the so-called Minkowski-Radström-Hörmander Theorem on embedding collections of closed, bounded, convex sets into a suitable Banach Space. However, an appeal to this embedding theorem seems not possible here to get results like those in our previous Theorem. This impossibility justify the use of semigroup-valued contents.

For our purpose of proving the Theorem stated above, let us introduce the complete metric semigroup $(\mathcal{K}, +, \{0\}, \delta)$ where $\mathcal{K}$ is the family of compact nonempty subsets of $\mathbf{R}^\ell$ and δ is the Hausdorff metric

$$\delta(A, B) = \inf \{t > 0 : \; A \subseteq B + C_t \text{ and } B \subseteq A + C_t\}.$$

It is evident that closed-valued, nonatomic f.a. set correspondences are nothing else than $\mathcal{K}-$valued nonatomic contents.

We need a few preparatory propositions.

Proposition 1. *If $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{K}$, then $\mathcal{A} \subseteq \overline{\mathcal{B}}^{\mathcal{K}}$ implies $\cup\mathcal{A} \subseteq \overline{\cup\mathcal{B}}$.*

Proposition 2. *Let (A_n) be a sequence in $\mathcal{K}$. If, in the semigroup $\mathcal{K}$, the series $\sum_n A_n$ has sum A, then:*

a) $x_n \in A_n \ \forall n \Rightarrow \sum_n x_n$ converges (uniformly with respect to the choice of $x_n \in A_n$);

b) $A = \{x : \ x = \sum_{i=1}^{\infty} x_i, \ x_i \in A_i\}$.

Proposition3. *Assume Φ is f.a. and bounded-valued. For E, $F \in \mathcal{F}$, the following are equivalent:*

a) $\Phi(F) \neq \{0\}$ and $E \subseteq F \Rightarrow \Phi(E) = \{0\}$ or $\Phi(F \setminus E) = \{0\}$;

b) $R(\Phi(\cdot \cap F)) \neq \{0\}$ and $E \subseteq F \Rightarrow R(\Phi(\cdot \cap E)) = \{0\}$ or
$R(\Phi(\cdot \cap F \setminus E)) = \{0\}$.

We are finally in a position to prove, quickly, our Theorem. Let X be the Stone space of $\mathcal{F}$ and $\mathcal{C}$ the algebra of closed-open subsets of X. Denote by Σ the σ-algebra generated by $\mathcal{C}$ and denote by h the Stone isomorphism. The position

$$m(hF) := \Phi(F)$$

gives a content $m : \mathcal{C} \to \mathcal{K}$ which is nonatomic, s-bounded and o-continuous. From Theorem B in the appendix, we extend m to Σ having the o-continuous content

$$\tilde{m} \colon \Sigma \to \mathcal{K}.$$

$\tilde{m}$ is nonatomic and (in $\mathcal{K}$) $R(\tilde{m}) \subseteq \overline{R(m)}$. But really $\tilde{m}$ is countably additive and atomless (appendix, Theorem A). From propositions 3 and 2 we see that $\tilde{m}$ is a "non-atomic set-valued measure" in the sense of [Ar] and therefore has convex values [Ar, Theorem 4.2], has convex range [Ar, Theorem 4.4] and [Ar, Theorem 8.3] for any $E \in \Sigma$ and $y \in \tilde{m}\ (E)$ there is a selection measure of $\tilde{m}$ that takes value y on E. To finish we use Proposition 1.□

An improvement of statement b) of our Theorem is possible if we assume that $\mathcal{F}$ is a σ-algebra:

Theorem (Lyapunov-type)[AB]. *A closed-valued, nonatomic, f.a. set correspondence, defined on a σ-algebra ha a convex range.*

APPENDIX. Let $(S, +, 0)$ be an abelian semigroup with neutral element 0. A pseudometric d on S is called *semiinvariant* if

$$d(a+c, b+c) \leq d(a,b) \qquad \text{for all } a, b, c \in S.$$

When a uniformity $\mathcal{U}$ on S is given such that it is generated by a family of seminivariant pseudometrics, then we say that $(S, +, 0, \mathcal{U})$ is a uniform semigroup. In particular S could be just endowed with one semiinvariant metric and in this case we should call it a metric semigroup. The semiinvariance guarantees the uniform continuity of the sum.

Assume now to deal with a separated and complete uniform semigroup S. Finitely additive set functions $\mu : \mathcal{F} \to S$ that are null on the empty set are here called ($S-$valued) contents. Let $\mu : \mathcal{F} \to \mathcal{S}$ be a content. We say that it is *s-bounded* (resp. *o-continuous*) if $\mu(E_n) \to 0$ whenever $E_n \in \mathcal{F}$ form a disjoint (resp. decreasing and with empty intersection) sequence; μ is called *nonatomic* if for any $U \in \mathcal{U}$ there exists in $\mathcal{F}$ a finite partition $\{F_1, \ldots, F_p\}$ of Ω such that

$$E \in \mathcal{F}, \quad E \subseteq F_i \Rightarrow (\mu(E), 0) \in U.$$

On the other hand we remind that an *atom* of μ is a set $F \in \mathcal{F} \setminus \mathcal{N}(\mu)$ such that

$$E \in \mathcal{F} \quad E \subseteq F \Rightarrow E \text{ or } F \setminus E \text{ belongs to } \mathcal{N}(\mu).$$

The results we need about ($S-$valued) contents are stated in the next two theorems. The proof of Theorem B is in [D] (a minor extra work is necessary). Theorem A is from [W].

Theorem A *If μ is o-continuous, $\mathcal{F}$ is a $\sigma-$algebra and S is metrizable, then*

$$\mu \textit{ nonatomic } \Leftrightarrow \mu \textit{ atomless.}$$

Theorem B*If the $S-$valued content μ is s-bounded, o- continuous and nonatomic, then there exists a unique o-continuous nonatomic $S-$valued content $\tilde{m}$ which extends μ to the $\sigma-$algebra generated by $\mathcal{F}$. Moreover the range of μ is dense in the range of $\tilde{m}$.*

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