"Napoleon's Theorem" and its Reciprocal.

W. Reyes

(0) Consider the Euclidean plane R^2 with origin 0 and provided with the usual inner product <,> and norm ‖,‖.

For any vector (point) $\overrightarrow{OA} = A$ let A' be the vector obtained from A through a positive 60^o rotation around 0; likewise, let A" and A"" be the vectors obtained from A through a 120^o and a 240^o rotation. Notice the following properties of these operators;

(i) $\langle A,B\rangle = \langle A',B\rangle + \langle A,B'\rangle$, $\|A\|^2 = \|A'\|^2 = 2\langle A,A'\rangle$;

(ii) $A'' = A' - A$, $A'''' = -A'$;

(iii) AA"A"" is a positively oriented equilateral triangle with side $\|A\|\sqrt{3}$ and centre 0.

Using the above properties we prove the so-called "Napoleon's Theorem" and also a version of it's reciprocal.

Napoleon's Theorem is the following statement.

(1) The centres of the equilateral triangles constructed, either outwards or inwards, over the sides of any triangle are the vertices of two concentric and differently oriented equilateral triangles. (The Napoleon triangles associated to the one given).

Proof. Consider a triangle ABC. For simplicity we may suppose A + B + C = 0. Let

$$X = (B - C)' + C, \quad Y = (C - A)' + A, \quad Z = (A - B)' + B.$$

Then BXC, CYA, AZB are precisely the exterior equilaterals over the sides. This is easy to verify:
$\|X - C\|^2 = \|B - C\|^2$. $\|X - B\|^2 = \|(B - C)' - B + C\|^2 = \|B - C\|^2$.

Let L, M, N be their respective centres, that is to say:

$$3L = (B - C)' + C - A, \quad 3M = (C - A)' + A - B,$$

$$3N = (A - B)' + B - C.$$

Now, $L'' = M$.

$[3L' = (B - C)'' + (C - A)' = (B - C)' - (B - C) + (C - A)' = (B - A)' - B + C;\ 3L'' = (B - A)'' - B' + C' = (B - A)' - B + A - B' + C' = (C - A)' + A - B = 3M]$.

Also $L'''' = N$, as it is verified in the same way. But all this implies, as in (0), (iii) that LMN is a positively oriented equilateral triangle with centre 0.

Moreover, the symmetricals L^*, M^*, N^* of L, M, N with respect to the corresponding sides are the centres of the inner equilateral triangles and are given by the relations

$$3L^* = (C - B)' + B - A, \quad 3M^* = (A - C)' + C - B,$$

$$3N^* = (B - A)' + A - C.$$

These vectors are subject to the following identities:

$$L^{*''} = N^*, \quad L^{*''''} = M^*,$$

which are verified as the former ones and which mean that $L^*N^*M^*$ is a positively oriented equilateral triangle with centre 0. So LMN and $L^*M^*N^*$ are the Napoleon triangles for ABC.

Now we proceed to the reciprocal situation, that is, to the following proposition.

(2) Let L and L* be such that $\|L\| > \|L^*\|$ and let LMN and L*M*N* be the equilateral triangles with centre 0 and sides $\|L\|\sqrt{3}$ and $\|L^*\|\sqrt{3}$ respectively. Suppose these triangles with different orientations.

Then, there is a unique triangle ABC whose Napoleon triangles are precisely LMN and L*M*N*.

Proof. From L and L* we construct the two further points

$$B = (L^* - L)' + L, \; C = (L - L^*)' + L^*.$$

Notice that L* and L are respectively the centres of CX*B and BXC, with $X^* = (C - B)' + B$, $X = (B - C)' + C$.

[In terms of L and L*, $X^* = 2L^* - L$, $X = 2L - L^*$, thus $B + C + X^* = L + L^* + 2L^* - L = 3L^*$, $B + C + X = 3L$].

But, because of relation (0) (ii) and because of the different orientations of LMN and L*M*N*, the following relations are valid:

$$M = L'' = L' - L, \; N = L'''' = -L',$$

$$M^* = L^{*''''} = -L^{*'}, \; N^* = L^{*''} = L^{*'} - L^*.$$

On account of the former we have still:

$(M - M^*)' + M^* = (L' - L + L^{*'})' - L^{*'} = (L + L^*)'' - (L + L^*)' = -(L + L^*)$.

$(M^* - M)' + M = (-L^{*'} - L' + L)' + L' - L = -(L^* + L)'' + 2L' - L = (L - L^*)' + L^* = C$.

$(N - N^*)' + N^* = (-L' - L^{*'} + L^*)' + L^{*'} - L^* = -(L^* + L)'' + 2L^{*'} - L^* = (L^* - L)' + L = B$.

$(N^* - N)' + N = (L^{*\prime} - L^* + L')' - L' = (L^* + L)'' - (L^* + L)'$

$= -(L + L^*)$.

Thus we have discovered a unique triangle ABC, with

$$A = -(L^* + L),\ B = (L^* - L)' + L,\ C = (L - L^*)' + L^*.$$

It has been already verified that L and L* are the centres of BXC and BX*C; a computation shows that M and M* are the centres of AYC and AY*C, also that N and N* are the centres of AZB and AZ*B, Y* and Z* being, of course, the symmetricals of Y and Z about the respective sides.

[For the outer triangles one verifies that

$$Y = (L^* + 2L)' - 2L,\ Z = -(2L + L^*)' + L^*,$$

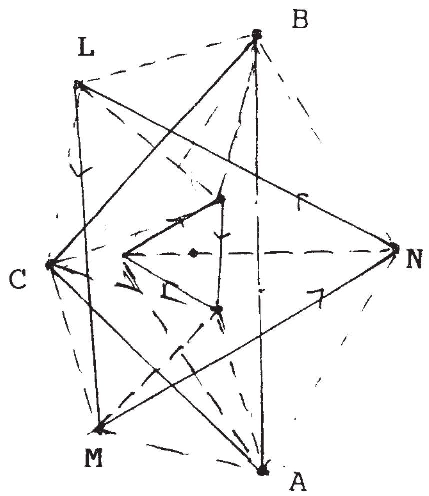
$$Y + A + C = (L^* + 2L)' - 2L - (L^* + L) + (L - L^*)' + L^* =$$

$$3(L' - L) = 3M.$$

$$Z + A + B = -(2L + L^*)' + L^* - (L^* + L) + (L^* - L)' =$$

$$-3L' = 3N].$$

<u>Note.</u> Although we do not rely on geometrical intuition, a picture may help to follow the proof.



Reference. For different proofs, plenty of remarks and many references, see

J. E. Wetzel, Converses of Napoleon's Theorem, The Am. Math. Monthly, April 1992.

U. Bio Bio, Chillan, Chile.