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$|3|$-GRADINGS OF COMPLEX CLASSICAL LIE ALGEBRAS

DIEGO LAGOS AND MAURICIO GODOY MOLINA

ABSTRACT. The aim of this note is to investigate the algebraic structure that appears on $|3|$-gradings $\mathfrak{n} = \mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ of a classical Lie algebra $\mathfrak{n}$ over $\mathbb{C}$. In particular, we prove that the negative part $\mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ of a grading can never be a free nilpotent Lie algebra of step 3, and completely determine the possible reductive algebras $\mathfrak{n}_0$ for Lie algebras of type A_n.

1. INTRODUCTION

A differential system is a pair (M,D), where M is a differentiable manifold and D is a distribution on M, that is, a subbundle of the tangent bundle of M. These objects appear naturally when studying certain problems related to constrained mechanics, where M is the configuration space of a mechanical system and D encodes a linear space of admissible velocities. There is a vast amount of literature regarding different points of view of these mathematical objects, since they play an important role in contact geometry [8], sub-Riemannian geometry [1, 9] and geometric control theory [7].

The study of symmetries of differential systems has been an important problem in differential geometry for over a century. For example, the seminal paper by É. Cartan [4] is nowadays understood as a complete study of the symmetries of differential systems with M a five-dimensional manifold and D of rank two. For the sake of context, let us recall that the group of global symmetries of a differential system (M,D) is

$$\mathrm{Sym}(M,D) = \{\varphi\colon M \to M \text{ diffeomorphism} \mid \varphi_* D = D\},$$

which is very difficult to determine in general. A well-known example is the Legendre transform in $\mathbb{R}^{2n+1}$, which is a global symmetry for the canonical contact structure D_{cont} (for details, see [8]). As usual in differential geometry, the infinitesimal object is easier to deal with, namely the Lie algebra of infinitesimal symmetries of (M,D), given by

$$\mathrm{sym}(M,D) = \{X \in \mathfrak{X}(M) \mid [X,\Gamma(D)] \subseteq \Gamma(D)\},$$

where $\Gamma(D)$ denotes the Lie algebra of sections of the distribution D. In this context, the infinitesimal symmetries of important differential systems can be found explicitly; for example, $\mathrm{sym}(\mathbb{R}^{2n+1}, D_{cont})$ is the infinite-dimensional jet space $J(\mathbb{R}^{2n+1})$.

The search for a way to determine the infinitesimal symmetries of special differential systems has proved fruitful over the years, especially as a consequence of the fundamental work by Tanaka [11], where an explicit linear algebraic procedure is given to determine $\mathrm{sym}(N,\mathfrak{n}_{-1})$ in the case where N is the (unique, up to isomorphism, connected and simply connected) nilpotent Lie group associated to a graded nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{-\mu} \oplus \cdots \oplus \mathfrak{n}_{-1}$. This process is referred to as *Tanaka prolongation*.

Using techniques from parabolic geometry (see [3]), the study of these very particular spaces of infinitesimal symmetries is related to $|s|$-gradings of semisimple Lie algebras.

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The aim of this short note is to provide some details in the case of $s=3$ for the classical Lie algebras. Additionally, from a different point of view, we can ask whether the nilpotent part of a $|3|$-grading is a Lie algebra of a certain kind (an idea exploited in [5] for the case of $|2|$-gradings). Here we present some preliminary results concerning the free nilpotent Lie algebras of step 3, which can be seen as a more concrete alternative to the general result obtained in [12]. Our complete study of $|3|$-gradings for all simple Lie algebras is in its final stage and will appear in print elsewhere.

2. Preliminaries

Let $\mathfrak{n}$ be a complex simple Lie algebra and $s \geq 1$ an integer. An $|s|$**-grading** of $\mathfrak{n}$ is a decomposition $\mathfrak{n} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_s$ such that:

(1) $[\mathfrak{n}_p, \mathfrak{n}_q] \subseteq \mathfrak{n}_{p+q}$, where we add that $\mathfrak{n}_p = \{0\}$ for $|p| > s$;
(2) the subalgebra $\mathfrak{n}_- = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_{-1}$ is generated by $\mathfrak{n}_{-1}$;
(3) $\mathfrak{n}_{-s} \neq \{0\}$ and $\mathfrak{n}_s \neq \{0\}$.

Remark 2.1. Given an $|s|$-grading $\mathfrak{n} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_s$, the term $\mathfrak{n}_0$ is a Lie subalgebra of $\mathfrak{n}$ and the Killing form of $\mathfrak{n}$ restricted to $\mathfrak{n}_i \times \mathfrak{n}_{-i}$ $(i = 1, \dots, s)$ is nondegenerate. In particular, we have $n_{-i} \cong n_i^*$ for all $i = 1, \dots, s$. For further details, see [3].

Let Δ be the set of roots of $\mathfrak{n}$ relative to a Cartan subalgebra $\mathfrak{h}$ and let $\Delta^0 = \{\alpha_1, \dots, \alpha_n\} \subseteq \Delta$ be the set of simple roots of Δ. Let $\Sigma \subseteq \Delta^0$ be a given subset of simple roots. For $\alpha = \sum a_i \alpha_i \in \Delta$, its Σ-height is

$$ht_\Sigma(\alpha) = \sum_{\alpha_i \in \Sigma} a_i.$$

If θ is the highest weight root of $\mathfrak{n}$, putting $s = ht_\Sigma(\theta)$ we define the $|s|$-grading of $\mathfrak{n}$ determined by Σ via

$$\mathfrak{n}_i = \bigoplus_{ht_\Sigma(\alpha)=i} \mathfrak{n}_\alpha \quad (i \neq 0) \qquad \text{and} \qquad \mathfrak{n}_0 = \mathfrak{h} \oplus \bigoplus_{ht_\Sigma(\alpha)=0} \mathfrak{n}_\alpha,$$

where $\mathfrak{n}_\alpha$ is the root space associated to the root α.

3. The $|3|$-gradings of complex classical Lie algebras

We start this section with a reinterpretation of Theorem 3.2.1 in [3].

Theorem 3.1. *Let $\mathfrak{n}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and set of simple roots Δ^0. Then, the $|s|$-gradings of $\mathfrak{n}$ are in bijection with the subsets $\Sigma \subseteq \Delta^0$ such that $ht_\Sigma(\theta) = s$, where θ is the highest root of $\mathfrak{n}$.*

The highest weight roots for the complex simple Lie algebras are well known and can be easily found in many textbooks, for example in [2]. For the ease of the reader, we summarize these roots for the classical Lie algebras in Table 1.

TABLE 1. Highest weight roots of the classical Lie algebras over $\mathbb{C}$.

Lie algebra	Highest weight root
$A_n, n \geq 1$	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
$B_n, n \geq 2$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n$
$C_n, n \geq 3$	$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + \alpha_n$
$D_n, n \geq 4$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$

Given a $|3|$-grading $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ of a simple Lie algebra $\mathfrak{n}$, we can compute the dimensions of $\mathfrak{n}_{-i}$ for $i = 0, 1, 2, 3$.

3.1. **The case A_n, $n \geq 4$.** Let $\mathfrak{n}$ be a Lie algebra of type A_n. Since the highest weight root of $\mathfrak{n}$ is $\alpha_1 + \cdots + \alpha_n$, Theorem 3.1 implies that the $|3|$-gradings of $\mathfrak{n}$ are in one-to-one correspondence with the subsets $\Sigma_{i,j,k} = \{\alpha_i, \alpha_j, \alpha_k\}$ of simple roots of $\mathfrak{n}$, where $1 \leq i < j < k \leq n$. Using the standard gradation given in [13] we can represent the $|3|$-grading $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ as follows:

$$\begin{array}{c|c|c|c|c} & i & j & k & \\ \hline & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline i & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline j & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline k & \mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 \end{array}$$

As a consequence we have the following proposition.

Proposition 3.2. *Let $\mathfrak{n}$ be a Lie algebra of type A_n. If $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ is the $|3|$-grading of $\mathfrak{n}$ determined by $\Sigma_{i,j,k}$, then*

$$\begin{aligned} \dim \mathfrak{n}_{-1} &= i(j-i) + (k-j)(j-i) + (n+1-k)(k-j), \\ \dim \mathfrak{n}_{-2} &= i(k-j) + (n+1-k)(j-i), \\ \dim \mathfrak{n}_{-3} &= i(n+1-k). \end{aligned}$$

3.2. **The case B_n.** Let $\mathfrak{n}$ be a Lie algebra of type B_n. The highest weight root of $\mathfrak{n}$ is $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$, and so the $|3|$-gradings $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ of $\mathfrak{n}$ are in one-to-one correspondence with the subsets $\Sigma_i = \{\alpha_1, \alpha_i\} \subseteq \Delta^0$, where $2 \leq i \leq n$. Using the standard gradation given in [13] we can represent the $|3|$-grading $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ as follows:

$$\begin{array}{c|c|c|c|c|c} & 1 & i & i & 1 & \\ \hline & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 & * \\ \hline 1 & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline i & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline i & \mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline 1 & * & \mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 \end{array}$$

As a consequence we have the following proposition.

Proposition 3.3. *Let $\mathfrak{n}$ be a Lie algebra of type B_n. If $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ is a $|3|$-grading of $\mathfrak{n}$, then*

$$\begin{aligned} \dim \mathfrak{n}_{-1} &= (i-1)(2n+2-2i), \\ \dim \mathfrak{n}_{-2} &= \tfrac{1}{2}(i-1)(i-2)+(2n+1-2i), \\ \dim \mathfrak{n}_{-3} &= i-1. \end{aligned}$$

3.3. **The case C_n.** Let $\mathfrak{n}$ be a Lie algebra of type C_n. The highest weight root of $\mathfrak{n}$ is $2\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$, and so the $|3|$-gradings $\mathfrak{n}_{-3}\oplus\cdots\oplus\mathfrak{n}_3$ of $\mathfrak{n}$ are in one-to-one correspondence with the subsets $\Sigma_i = \{\alpha_i, \alpha_n\} \subseteq \Delta^0$, where $1 \le i \le n-1$. Using the standard gradation given in [13] we can represent the $|3|$-grading $\mathfrak{n}_{-3}\oplus\cdots\oplus\mathfrak{n}_3$ as follows:

$$\begin{array}{c} \qquad\qquad\quad i \qquad\quad n \qquad\quad i \\ \begin{array}{c} \\ i \\ \\ n \\ \\ i \\ \\ \end{array} \left(\begin{array}{c|c|c|c} \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline \mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 \end{array}\right) \end{array}$$

As a consequence we have the following proposition.

Proposition 3.4. *Let $\mathfrak{n}$ be a Lie algebra of type C_n. If $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ is a $|3|$-grading of $\mathfrak{n}$, then*

$$\begin{aligned} \dim \mathfrak{n}_{-1} &= i(n-i)+(n-i)+\tfrac{1}{2}(n-i)(n-i-1), \\ \dim \mathfrak{n}_{-2} &= i(n-i), \\ \dim \mathfrak{n}_{-3} &= i+\tfrac{1}{2}i(i-1). \end{aligned}$$

3.4. **The case D_n.** Let $\mathfrak{n}$ be a Lie algebra of type D_n. The highest weight root of $\mathfrak{n}$ is $\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$, and so the $|3|$-gradings of $\mathfrak{n}$ are in one-to-one correspondence with the the following subsets of simple roots of $\mathfrak{n}$:

(1) $\Sigma_{i,1} = \{\alpha_1, \alpha_i\}$,
(2) $\Sigma_{i,n} = \{\alpha_i, \alpha_n\}$,
(3) $\Sigma_{i,n-1} = \{\alpha_i, \alpha_{n-1}\}$,
(4) $\Sigma_{1,n-1,n} = \{\alpha_1, \alpha_{n-1}, \alpha_n\}$,

where $2 \le i \le n-2$.

There exists an automorphism of the Dynkin diagram that permutes α_{n-1} and α_n, and thus there exists an automorphism of D_n giving an isomorphism between the gradings induced by $\Sigma_{i,n-1}$ and $\Sigma_{i,n}$ (see [6, Chapter 14]). Therefore, in the following proposition we only consider the sets $\Sigma_{i,1} = \{\alpha_1, \alpha_i\}$, $\Sigma_{i,n} = \{\alpha_i, \alpha_n\}$ and $\Sigma_{1,n-1,n} = \{\alpha_1, \alpha_{n-1}, \alpha_n\}$. Using the standard gradation given in [13] we have:

- For $\Sigma_{i,1}$ the |3|-grading is given by

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & i & i & 1 \end{array}\\
\begin{array}{c} 1 \\ i \\ i \\ 1 \end{array}
\left(\begin{array}{c|c|c|c|c}
\mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 & * \\ \hline
\mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline
\mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline
\mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline
* & \mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0
\end{array}\right).
\end{array}
$$

- For $\Sigma_{i,n}$ the |3|-grading is given by

$$
\begin{array}{c}
\begin{array}{ccc} i & n & i \end{array}\\
\begin{array}{c} i \\ n \\ i \end{array}
\left(\begin{array}{c|c|c|c}
\mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline
\mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline
\mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline
\mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0
\end{array}\right).
\end{array}
$$

- For $\Sigma_{1,n-1,n}$ the |3|-grading is given by

$$
\begin{array}{c}
\begin{array}{ccc} n-1 & n & n-1 \end{array}\\
\begin{array}{c} n-1 \\ n \\ n-1 \end{array}
\left(\begin{array}{c|c|c|c}
\mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 & \mathfrak{n}_3 \\ \hline
\mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 & \mathfrak{n}_2 \\ \hline
\mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0 & \mathfrak{n}_1 \\ \hline
\mathfrak{n}_{-3} & \mathfrak{n}_{-2} & \mathfrak{n}_{-1} & \mathfrak{n}_0
\end{array}\right).
\end{array}
$$

As a consequence we have the following proposition.

Proposition 3.5. *Let $\mathfrak{n}$ be a Lie algebra of type D_n and let $\mathfrak{n}_{-3}\oplus\cdots\oplus\mathfrak{n}_3$ be a $|3|$-grading of $\mathfrak{n}$ associated to the previous subsets of simple roots of $\mathfrak{n}$. Then*

(1) *For $\Sigma_{i,1}$ we have*

$$\begin{aligned}\dim\mathfrak{n}_{-1} &= (2n-2i)(i-1)+i-1,\\ \dim\mathfrak{n}_{-2} &= \tfrac{1}{2}(i-1)(i-2)+(2n-2i),\\ \dim\mathfrak{n}_{-3} &= i-1.\end{aligned}$$

(2) *For $\Sigma_{i,n}$ we have*

$$\begin{aligned}\dim\mathfrak{n}_{-1} &= \tfrac{1}{2}(n-i)(n-i-1)+i(n-i),\\ \dim\mathfrak{n}_{-2} &= i(n-i),\\ \dim\mathfrak{n}_{-3} &= \tfrac{1}{2}i(i-1).\end{aligned}$$

(3) *For $\Sigma_{1,n-1,n}$ we have*

$$\begin{aligned}\dim\mathfrak{n}_{-1} &= 3(n-2),\\ \dim\mathfrak{n}_{-2} &= 2+\tfrac{1}{2}(n-2)(n-3),\\ \dim\mathfrak{n}_{-3} &= n-2.\end{aligned}$$

Remark 3.6. In the case of the exceptional Lie algebras, it is also possible to compute all of these dimensions for all $|3|$-gradings. To do this, we use the fact that $\mathfrak{n}_i\cong\mathfrak{n}_{-i}$ and

$$\mathfrak{n}_i=\bigoplus_{ht_\Sigma(\alpha)=i}\mathfrak{g}_\alpha \quad (i=1,2,3),$$

where the sum is over all positive roots α with $ht_\Sigma(\alpha)=i$.

4. Free nilpotent Lie algebras

An important class of nilpotent Lie algebras is given by the free nilpotent Lie algebras of step $s\geq 2$ with r generators. These algebras have a very rich combinatorial structure and they are relevant in many areas of mathematics. For a classical in-depth introduction to the subject, see [10].

Definition 4.1. The *free nilpotent Lie algebra of step $s\geq 2$ with r generators* is the Lie algebra generated by a set $\mathscr{A}=\{x_1,\dots,x_r\}$ where there are no relations between the x_i's except for the Jacobi identity and the condition that all brackets of order $\geq s$ are zero. This algebra is denoted by $\mathfrak{F}_{r,s}$.

Example 4.2. The free nilpotent Lie algebra $\mathfrak{F}_{2,2}$ is the Heisenberg algebra of dimension 3.

Any free nilpotent Lie algebra $\mathfrak{F}_{r,s}$, with $r\geq 2$ and $s\geq 2$, is naturally endowed with a Lie algebra grading

$$\mathfrak{F}_{r,s}=\mathfrak{f}_{-s}\oplus\cdots\oplus\mathfrak{f}_{-1},$$

where $\mathfrak{f}_{-1}$ is the span of $\mathscr{A}$ and $\mathfrak{f}_{-j}$ is spanned by the Lie brackets of order j. Moreover, the dimension of each $\mathfrak{f}_{-j}$ is given by

$$\dim\mathfrak{f}_{-j}=\frac{1}{j}\sum_{d|j}\mu(d)r^{j/d},$$

where μ is the Möbius function. The proof of this fact can be found in [10, Chapter 4].

In particular, for $s = 3$ we have

$$\dim \mathfrak{f}_{-1} = r,$$
$$\dim \mathfrak{f}_{-2} = \tfrac{1}{2} r(r-1),$$
$$\dim \mathfrak{f}_{-3} = \tfrac{1}{3}(r^3 - r).$$

5. |3|-GRADINGS AND FREE NILPOTENT LIE ALGEBRAS

In this section, $\mathfrak{n}$ denotes a complex simple Lie algebra of type A_n, B_n, C_n or D_n.

Definition 5.1. A *graded isomorphism* between $\mathfrak{F}_{r,3} = \mathfrak{f}_{-3} \oplus \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1}$ and the negative part $\mathfrak{n}_- = \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ of a $|3|$-grading of $\mathfrak{n}$ is a Lie algebra isomorphism

$$\phi : \mathfrak{f}_{-3} \oplus \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1} \to \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$$

such that $\phi|_{\mathfrak{f}_{-i}}$ is a linear isomorphism between $\mathfrak{f}_{-i}$ and $\mathfrak{n}_{-i}$ for $i = 1, 2, 3$.

Theorem 5.2. *Let $\mathfrak{n}$ be a $|3|$-grading of a simple Lie algebra of type B_n, C_n or D_n, and let $\mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ be a $|3|$-grading of $\mathfrak{n}$. Then there is no graded isomorphism between $\mathfrak{F}_{r,3}$ and $\mathfrak{n}_- = \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$.*

Proof for the case B_n. Suppose that there exists a graded isomorphism $\phi : \mathfrak{F}_{r,3} \to \mathfrak{n}_-$. Then, Proposition 3.3 implies that

$$r = (i-1)(2n+2-2i), \tag{1}$$

$$\frac{r(r-1)}{2} = \frac{(i-1)(i-2)}{2} + (2n+1-2i), \tag{2}$$

$$\frac{r^3 - r}{3} = i - 1. \tag{3}$$

Combining equalities (1), (2) and (3), we obtain

$$\frac{r(r-1)}{2} = \frac{1}{2}\left(\frac{r^3-r}{3} - 1\right)\frac{r^3-r}{3} + 2n + 1 - 2\left(\frac{r^3-r}{3} + 1\right),$$

which implies that

$$-r^6 + 2r^4 + 15r^3 + 8r^2 - 24r = 36n - 18.$$

For $r \geq 3$ the function $f(r) = -r^6 + 2r^4 + 15r^3 + 8r^2 - 24r$ is negative. For $r = 2$ the function f takes the value 72, which implies that $36n = 90$. This contradicts the assumption that n is an integer.

It follows that there cannot exist graded isomorphisms for $\mathfrak{n}$ of type B_n. □

Remark 5.3. For the Lie algebras of type C_n and D_n stated in Theorem 5.2, the arguments for the nonexistence of graded isomorphisms are also based in simple arithmetic, although they are computationally more involved.

A similar result for the case of Lie algebras of type A_n is also valid, but follows from a different argument. To show it, let us recall the following very simple fact.

Lemma 5.4. *Denote by E_{pq} the matrix with a 1 in the position (p,q) and 0's in all other entries. Then*

$$[E_{pq}, E_{rs}] = E_{pq}E_{rs} - E_{rs}E_{pq} = \begin{cases} E_{pq} & \text{if } p \neq s,\ q = r, \\ -E_{qr} & \text{if } p = s,\ q \neq r, \\ E_{pp} - E_{qq} & \text{if } p = s,\ q = r, \\ 0 & \text{if } p \neq s,\ q \neq r. \end{cases}$$

And now we can state and prove a result analogous to Theorem 5.2. This completes the nonexistence of graded isomorphisms for all classical simple Lie algebras.

Theorem 5.5. *Let $\mathfrak{n} = \mathfrak{n}_{-3} \oplus \cdots \oplus \mathfrak{n}_3$ be a $|3|$-grading of a simple Lie algebra $\mathfrak{n}$ of type A_n. Then there does not exist a graded isomorphism between $\mathfrak{F}_{r,3}$ and $\mathfrak{n}_- = \mathfrak{n}_{-3} \oplus \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$.*

Proof. We will first show that there are always nontrivial elements $x, y \in \mathfrak{n}_{-1}$ such that $[x,y] = 0$.

Recall that $\mathfrak{n}_{-1}$ is described as follows:

$$\mathfrak{n}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \end{pmatrix} : A \in M_{(j-i)\times i}, B \in M_{(k-j)\times(j-i)}, C \in M_{(n+1-k)\times(k-j)} \right\}.$$

Consider $x = E_{pq}$ and $y = E_{rs}$, with $p \in \{i+1, \ldots, j\}$, $q \in \{1, \ldots, i\}$, $r \in \{k+1, \ldots, n+1\}$ and $s \in \{j+1, \ldots, k\}$. Then it is clear that $p \neq s$ and $q \neq r$. Applying Lemma 5.4 we see that these elements satisfy $[x,y] = 0$.

This obviously implies that a graded isomorphism between $\mathfrak{F}_{r,3}$ and $\mathfrak{n}_-$ cannot exist, since $[x,y]$ should be the image of a nontrivial element of $\mathfrak{f}_{-2}$. ☐

6. Further results

We have also been able to describe precisely the subalgebras $\mathfrak{n}_0$ for all $|3|$-gradings of a simple Lie algebra $\mathfrak{n}$. These algebras are known to be reductive and their classification follows from the next result, which can be found in [3].

Theorem 6.1. *Let $\mathfrak{n} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_s$ be an $|s|$-grading of a simple Lie algebra associated to a subset Σ of simple roots of $\mathfrak{n}$. The dimension of the center of $\mathfrak{n}_0$ coincides with the cardinality of Σ, and the Dynkin diagram of the semisimple part $\mathfrak{n}_0^{ss}$ of $\mathfrak{n}_0$ is obtained by removing all nodes corresponding to the roots in Σ and all edges connected to these nodes.*

Using Theorem 3.1 and Theorem 6.1 we obtain Table 2, which shows the structure of $\mathfrak{n}_0$ for all possible $|3|$-gradings of a simple Lie algebra $\mathfrak{n}$ of type A_n.

TABLE 2. $\mathfrak{n}_0$ for algebras of type A_n.

Σ	$\mathfrak{n}_0$
$\{\alpha_1, \alpha_2, \alpha_3\}$	$\mathbb{C}^3 \oplus A_{n-3}$
$\{\alpha_1, \alpha_2, \alpha_n\}$	$\mathbb{C}^3 \oplus A_{n-3}$
$\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}$, $2 \leq i < n-3$	$\mathbb{C}^3 \oplus A_{i-1} \oplus A_{n-i-2}$
$\{\alpha_i, \alpha_{i+1}, \alpha_k\}$, $2 \leq i \leq n-3$, $\lvert k-(i+1)\rvert > 1$, $k \leq n-1$	$\mathbb{C}^3 \oplus A_{i-1} \oplus A_{k-i-2} \oplus A_{n-k}$
$\{\alpha_i, \alpha_{i+1}, \alpha_n\}$	$\mathbb{C}^3 \oplus A_{i-1} \oplus A_{n-i-2}$
$\{\alpha_i, \alpha_j, \alpha_k\}$, $\lvert j-i\rvert > 1$, $\lvert k-j\rvert > 1$, $k \leq n-1$	$\mathbb{C}^3 \oplus A_{i-1} \oplus A_{j-i-1} \oplus A_{k-j-1} \oplus A_{n-k}$
$\{\alpha_i, \alpha_j, \alpha_n\}$	$\mathbb{C}^3 \oplus A_{i-1} \oplus A_{j-i-1} \oplus A_{n-j-1}$

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(Diego Lagos) Universidad de la Frontera, Departamento de Matemática y Estadística, Temuco, Chile

Email address: diego.lagos@ufrontera.cl

(Mauricio Godoy Molina) Universidad de la Frontera, Departamento de Matemática y Estadística, Temuco, Chile

Email address: mauricio.godoy@ufrontera.cl