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WEAK*-CLOSURE OF CERTAIN SUBSPACES OF THE DUAL OF SOME ABELIAN BANACH ALGEBRAS

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ABSTRACT. We shall determine the weak*-closure of some subspaces of the dual of certain Banach algebras. In particular, we consider this problematic in the context of abelian C^*-algebras.

1. INTRODUCTION

Let I be a closed ideal of an abelian Banach algebra $\mathscr{A}$ with bounded approximate identity. The determination of conditions under which I itself is provided with a bounded approximate identity is a matter of interest. Recently, it was shown how certain idempotents of the second dual space $\mathscr{A}^{**}$ of $\mathscr{A}$ are related with this problem. In particular, $\mathscr{A}^{**}$ is considered with the Banach algebra structure given by any one of the two canonical Arens products, in such a way that $\mathscr{A}$ becomes a Banach subalgebra of $\mathscr{A}^{**}$ by means of the natural isometric immersion $\chi_{\mathscr{A}} : \mathscr{A} \hookrightarrow \mathscr{A}^{**}$ [1]. In this framework, I has a bounded approximate identity if and only if there is an idempotent $a^{**} \in \mathscr{A}^{**}$ so that the space $a^{**}\mathscr{A}^{*}$ is weak*-closed in $\mathscr{A}^{*}$ and $I = \{a \in \mathscr{A} : aa^{**} = 0\}$ (cf. [8, Lemma 2.3]). Consequently, the characterization of elements $a^{**} \in \mathscr{A}^{**}$ so that $a^{**}\mathscr{A}^{*}$ is weak*-closed is an issue that deserves consideration and probably is interesting on its own.

2. ABOUT THE WEAK*-CLOSEDNESS OF $a^{**}\mathscr{A}^{*}$

Theorem 1. *Given $a^{**} \in \mathscr{A}^{**}$ let $\rho_{a^{**}} : \mathscr{A} \to \mathscr{A}^{**}$ be the map $\rho_{a^{**}}(a) = aa^{**}$ if $a \in \mathscr{A}$.*

(1) *The following equalities hold:*

$$\begin{aligned}(a^{**}\mathscr{A}^{*})^{-w^{*}} &= \cap\{\ker \chi_{\mathscr{A}}(a) : a \in {}^{\perp}(a^{**}\mathscr{A}^{*})\} \\ &= \cap\{\ker \chi_{\mathscr{A}}(a) : aa^{**} = 0_{\mathscr{A}^{**}}\} \\ &= (\ker(\rho_{a^{**}}))^{\perp}.\end{aligned}$$

(2) *Let $\mathscr{A}$ be a weakly compact Banach algebra. Then $a^{**}\mathscr{A}^{*}$ is weak*-closed in $\mathscr{A}^{*}$ if and only if $\Re[(\chi_{\mathscr{A}})^{-1} \circ \rho_{a^{**}}]$ is closed in $\mathscr{A}$.*

Proof. (1) As the weak*-topology is locally convex by the Hahn–Banach separation theorem is $(a^{**}\mathscr{A}^{**})^{-w^{*}} = \cap_{\gamma\in\Gamma} \ker(\gamma)$, where Γ denotes the set of weak*-continuous linear forms that annihilates on $a^{**}\mathscr{A}^{**}$ (cf. [6, Cor. 1.2.13]). Moreover, any weak*-continuous linear form γ is realized as an evaluation, i.e., $\gamma \in \Im(\chi_{\mathscr{A}})$ (cf. [6, Prop. 1.3.5]). The second and third equalities are immediate.

(2) Since $\mathscr{A}$ is weakly compact $\chi_{\mathscr{A}}(\mathscr{A})$ becomes a two sided ideal of $\mathscr{A}^{**}$ [11]. So $(\chi_{\mathscr{A}})^{-1} \circ \rho_{a^{**}} \in B(A)$ and $a^{**}x^{*} = ((\chi_{\mathscr{A}})^{-1} \circ \rho_{a^{**}})^{*}(x^{*})$ for all $x^{*} \in \mathscr{A}^{*}$. Now the assertion follows by ([3, Ch. VI, Th. 1.10]).

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□

Corollary 1. *If $a^{**} \in \mathscr{A}^{**}$, $a^{**}\mathscr{A}^*$ is weak*-closed if and only if* $(\ker(\rho_{a^{**}}))^\perp \subseteq a^{**}\mathscr{A}^*$.

Example 1. Let $\mathscr{A} = c_0(S)$ be the usual Banach algebra of continuous functions vanishing at infinity on an infinite discrete set S. Then $\mathscr{A}^* \approx l^1(S)$ and $\mathscr{A}^{**} \approx l^\infty(S)$. Given $a^{**} \in \mathscr{A}^{**}$ let us write $\sigma(a^{**}) = \{s \in S : a_s^{**} \neq 0\}$. For $s \in S$ let $\delta^s \in \mathscr{A}^*$ be the current delta function on S so that $\delta_s(s) = 1$ and $\delta_s(t) = 0$ if $t \in S - \{s\}$. By Theorem 1 and as $\{\delta^s\}_{s\in S}$ is a basis of $\mathscr{A}^*$ it is easy to see that

$$(a^{**}\mathscr{A}^*)^{-w^*} = \mathrm{span}[\delta^s : s \in \sigma(a^{**})]^{-w^*} = \mathrm{span}[\delta^s : s \in \sigma(a^{**})]^-.$$

So $a^{**}\mathscr{A}^*$ is weak*-closed if and only if $\sum_{s\in\sigma(a^{**})} \mid a_s^*/a_s^{**} \mid < \infty$ whenever $\sum_{s\in\sigma(a^{**})} \mid a_s^* \mid < \infty$. This condition holds if $\sigma(a^{**})$ is finite. Otherwise, let $P_f(\sigma(a^{**}))$ be the class of finite subsets of $\sigma(a^{**})$ and if $F \in P_f(\sigma(a^{**}))$ let $T_F \in l^1(\sigma(a^{**}))^*$ so that $T_F(a^*) = \sum_{s\in F} a_s^*/a_s^{**}$. By the uniform boundedness principle the class $\{T_F\}_{F\in P_f(\sigma(a^{**}))}$ becomes bounded. But $\| T_F \| = \max\{\mid a_s^{**} \mid^{-1} : s \in F\}$ for each F. Consequently, $a^{**}\mathscr{A}^*$ is weak*-closed if and only if $\inf \sigma(a^{**}) > 0$.

3. When $\mathscr{A}$ is an abelian C^*-algebra

Theorem 2. *Let $\mathscr{A}$ be an abelian non-reflexive C^*-algebra and let $a^{**} \in \mathscr{A}$.*

(1) *If a^{**} is invertible or quasi-nilpotent then $a^{**}\mathscr{A}^*$ is weak*-closed.*
(2) *Let $a^{**} \in \mathscr{A}^{**} - \chi_{\mathscr{A}}(\mathscr{A})$ be idempotent.*
   (a) *There exists a weak*-dense norm-closed subspace $\Sigma_{a^{**}}$ of* $\ker(\rho_{a^{**}})^\perp$ *so that $\mathscr{A}a^{**} \subseteq \chi_{\mathscr{A}}(\mathscr{A}) \oplus [\Sigma_{a^{**}}]^\perp$.*
   (b) *Let $\{a_t\}_{t\in T}$ be a bounded net of $\mathscr{A}$ so that $a^{**} = w^* - \lim_{t\in T} \chi_{\mathscr{A}}(a_t)$. The set $\mathscr{J} = \{a \in \mathscr{A} : \exists x \in \mathscr{A} / a = w - \lim_{t\in T}(xa_t)\}$ is an ideal of $\mathscr{A}$. Further, given $a \in \mathscr{A}$, $\chi_{\mathscr{A}}(a) \in \mathscr{A}a^{**}$ if and only if $a \in \mathscr{J}$.*
   (c) *$a^{**}\mathscr{A}^*$ is weak*-closed if and only if $a^{**}\mathscr{A}^* = \mathscr{A}^*$.*

Proof. Since $\mathscr{A}$ is a complex abelian C^*-algebra it becomes Arens regular [10]. Indeed, the second conjugate algebra $(\mathscr{A}^{**}, \Box)$ becomes a C^*-algebra that is abelian because $\mathscr{A}$ is abelian and regular (cf. [2, Th. 7.1]). Besides $\mathscr{A}$ has a bounded approximate identity and so $(\mathscr{A}^{**}, \Box)$ is unital. If $\Delta(\mathscr{A}^{**})$ denotes the maximal ideal space of $(\mathscr{A}^{**}, \Box)$ the Gelfand transform $G : (\mathscr{A}^{**}, \Box) \to C(\Delta(\mathscr{A}^{**}))$ provides an isometric isomorphism of Banach algebras.

(1) Given $a^{**} \in \mathscr{A}^{**}$ let $a \in \ker(\rho_{a^{**}})$. Then

$$0_{C(\Delta(\mathscr{A}^{**}))} = G(aa^{**}) = G(\chi_{\mathscr{A}}(a))G(a^{**}),$$

i.e., $\mathfrak{h}(\chi_{\mathscr{A}}(a))\mathfrak{h}(a^{**}) = 0$ for all $\mathfrak{h} \in \Delta(\mathscr{A}^{**})$.

If a^{**} is invertible it is clear that $\sigma_{\mathscr{A}^{**}}(\chi_{\mathscr{A}}(a)) = \{0\}$ and $\sigma_{\mathscr{A}}(a) = \{0\}$ because $\chi_{\mathscr{A}}$ is isometric. So it is readily seen that $a = 0_{\mathscr{A}}$ and $(\ker(\rho_{a^{**}}))^\perp = \mathscr{A}^*$.

If a^{**} is quasi-nilpotent $\ker(\rho_{a^{**}}) = \mathscr{A}$ and $(\ker(\rho_{a^{**}}))^\perp = \{0_{\mathscr{A}^*}\}$.

In both cases it is plain that $a^{**}\mathscr{A}^*$ becomes weak*-closed if $a^{**} \in \mathscr{A}^{**}$ is invertible or quasi-nilpotent and the claim follows.

(2) Let us write $I(a^{**}) = G(a^{**})^{-1}[\sigma_{\mathscr{A}^{**}}(a^{**}) - \{0\}]$. We have

$$\ker(\rho_{a^{**}}) = \cap\{\ker(\mathfrak{h} \circ \chi_{\mathscr{A}}) : \mathfrak{h} \in I(a^{**})\}.$$

Therefore

$$(\ker(\rho_{a^{**}}))^\perp = \mathrm{span}[\cup_{\mathfrak{h}\in I(a^{**})} \ker(\mathfrak{h} \circ \chi_{\mathscr{A}})^\perp]^{-w^*} = [\Sigma_{a^{**}}]^{-w^*}, \tag{1}$$

with $\Sigma_{a^{**}} = \mathrm{span}[S_{a^{**}}]^{-}$ and $S_{a^{**}} = \{\mathfrak{h} \circ \chi_{\mathscr{A}} : \mathfrak{h} \in I(a^{**})\}$. If $\mathfrak{h} \in \Delta(\mathscr{A}^{**})$ it is straightforward to see that $a^{**}(\mathfrak{h} \circ \chi_{\mathscr{A}}) = \langle \mathfrak{h} \circ \chi_{\mathscr{A}}, a^{**} \rangle \mathfrak{h} \circ \chi_{\mathscr{A}}$. Further, let $\mathfrak{h} \in I(a^{**})$ so that $\langle \mathfrak{h} \circ \chi_{\mathscr{A}}, a^{**} \rangle \neq 0$. If $a^{**}\mathscr{A}^{*}$ is weak*-closed we can write $\mathfrak{h} \circ \chi_{\mathscr{A}} = a^{**}x^{*}_{\mathfrak{h}}$ for some $x^{*}_{\mathfrak{h}} \in \mathscr{A}^{*}$.

(a) If a^{**} is idempotent we have

$$a^{**}x^{*}_{\mathfrak{h}} = a^{**}(\mathfrak{h} \circ \chi_{\mathscr{A}}) = \langle \mathfrak{h} \circ \chi_{\mathscr{A}}, a^{**} \rangle \mathfrak{h} \circ \chi_{\mathscr{A}} = \mathfrak{h} \circ \chi_{\mathscr{A}}.$$

Then $\langle \mathfrak{h} \circ \chi_{\mathscr{A}}, a^{**} \rangle = 1$ and $a^{**}(\mathfrak{h} \circ \chi_{\mathscr{A}}) = \mathfrak{h} \circ \chi_{\mathscr{A}}$. Thus, if $a \in \mathscr{A}$ and $a^{*} \in \Sigma_{a^{**}}$ we see that

$$\langle a^{*}, \chi_{\mathscr{A}}(a) \rangle = \langle a, a^{*} \rangle = \langle a, a^{**}a^{*} \rangle = \langle a^{*}a, a^{**} \rangle = \langle a^{*}, \rho_{a^{**}}(a) \rangle,$$

i.e., $\rho_{a^{**}}(a) - \chi_{\mathscr{A}}(a) \in [\Sigma_{a^{**}}]^{\perp}$. If $\chi_{\mathscr{A}}(a) \in [\Sigma_{a^{**}}]^{\perp}$ let us consider a nonzero homomorphism $\varphi_0 : \mathscr{A} \to \mathbb{C}$. It admits a natural extension, on the C^{*}-subalgebra $\mathscr{Q}$ of $\mathscr{A}^{**}$ generated by $\chi_{\mathscr{A}}(\mathscr{A}) \cup \{a^{**}\}$, to an homomorphism φ_1 such that $\varphi_1(a^{**}) = 1$. Since $\mathscr{Q}$ is a commutative symmetric Banach *-algebra its Shilov boundary $\partial\mathscr{Q}$ coincides with the whole maximal ideal space $\Delta(\mathscr{Q})$ (cf. [7, Example 3.3.16]). Thus φ_1 has an extension to a character $\varphi_2 \in \mathscr{A}^{**}$ (cf. [4, Cor. 1]). Hence $\varphi_2 \in I(a^{**})$, $\varphi_2 \circ \chi_{\mathscr{A}} \in \Sigma_{a^{**}}$ and

$$\begin{aligned} 0 &= \langle \varphi_2 \circ \chi_{\mathscr{A}}, \chi_{\mathscr{A}}(a) \rangle \\ &= \langle a, \varphi_2 \circ \chi_{\mathscr{A}} \rangle \\ &= \langle \chi_{\mathscr{A}}(a), \varphi_2 \rangle \\ &= \langle \chi_{\mathscr{A}}(a), \varphi_1 \rangle \\ &= \langle a, \varphi_0 \rangle. \end{aligned}$$

Thus $a \in \mathscr{A}$ must be quasi-nilpotent and we can conclude that $a = 0_{\mathscr{A}}$, i.e., $\mathscr{A}a^{**} \subseteq \chi_{\mathscr{A}}(\mathscr{A}) \oplus [\Sigma_{a^{**}}]^{\perp}$.

(b) It is easy to see that $\mathscr{J}$ is an ideal of $\mathscr{A}$, eventually trivial. Let $a \in \mathscr{A}$ so that $\chi_{\mathscr{A}}(a) = xa^{**}$ for some $x \in \mathscr{A}$. Given $a^{*} \in \mathscr{A}^{*}$ we have

$$\langle a, a^{*} \rangle = \langle a^{*}x, a^{**} \rangle = \lim_{t \in T} \langle a_t, a^{*}x \rangle = \langle xa_t, a^{*} \rangle,$$

i.e., $a \in \mathscr{J}$. Likewise, if $a \in \mathscr{J}$ and $a = w - \lim_{t \in T}(xa_t)$ for some $x \in \mathscr{A}$ then

$$\langle a^{*}, xa^{**} \rangle = \lim_{t \in T} \langle a_t, a^{*}x \rangle = \langle a, a^{*} \rangle$$

and so $\chi_{\mathscr{A}}(a) = xa^{**}$.

(c) Since $(\ker(\rho_{a^{**}}))^{\perp\perp} = [\chi_{\mathscr{A}}(\ker(\rho_{a^{**}}))]^{-w^{*}}$ by (1) we see that

$$\chi_{\mathscr{A}}(\ker(\rho_{a^{**}})) \subseteq [\Sigma_{a^{**}}]^{\perp}.$$

Consequently $\rho_{a^{**}}$ must be injective and the assertion follows by Corollary 1.

□

Example 2. Let $\mathscr{A} = C_0(G)$ be the uniform Banach algebra of complex functions vanishing at infinity on a locally compact abelian group G with Haar invariant measure λ and identity element e. There is an isometric isomorphism of Banach spaces between $\mathscr{A}^{*}$ and the Banach space $M(G)$ of complex bounded regular Borel measures (cf. [9, Th. 2.14]). Further, by the Lebesgue–Radon–Nikodym decomposition theorem,

$$\mathscr{A}^{*} \approx l^{1}(G) \oplus_1 L^{1}(G, \lambda) \oplus_1 M_{cs}(G, \lambda),$$

i.e., any element of $M(G)$ can be represented uniquely as the sum of a discrete measure, an absolutely continuous and a singular continuous measure with respect to λ (cf. [9, Th. 6.10]). Consequently,

$$\mathscr{A}^{**} \approx l^{\infty}(G) \oplus_{\infty} L^{\infty}(G) \oplus_{\infty} M_{sc}(G,\lambda)^{*}.$$

Let $m,n \in l^{\infty}(G)$, $M,N \in L^{\infty}(G)$ and $\mu,\eta \in M_{sc}(G)^{*}$. It is straightforward to see that

$$(m,M,\mu)(n,N,\eta) = (mn, MN, \mu \square \eta),$$

where $\square$ is the current Arens product of $\mathscr{A}^{**}$ and mn and MN represent the pointwise products of $l^{\infty}(G)$ and $L^{\infty}(G)$. Let W be a compact neighbourhood of e and let $M = I_{W-\{e\}}$, where $I_{W-\{e\}}$ is the usual characteristic function of $W-\{e\}$. Then M can be viewed as an idempotent of $\mathscr{A}^{**}$ and $M\mathscr{A}^{*}$ is not weak*-closed. For instance, let $\mathscr{N} = \{\lambda(U)^{-1} I_U\}_{U \in \mathscr{U}_e}$, where $\mathscr{U}_e$ is the directed set of relatively compact symmetric neighbourhoods of e. The net $\mathscr{N}$ is bounded in $L^1(G)$ and $\delta_e = w^* - \lim_{U \in \mathscr{U}_e} F[\lambda(U)^{-1} I_U]$, i.e., $\delta_e \in (M\mathscr{A}^{*})^{-w^*}$. Nevertheless, $M\mathscr{A}^{*} = Ml^1(G) \oplus_1 ML^1(G,\lambda) \oplus_1 (0_{M_{cs}(G,\lambda)^*})$ and

$$Ml^1(G) = \{\zeta \in l^1(G) : \operatorname{supp}(\zeta) \subseteq W - \{e\}\}$$

and therefore $M\mathscr{A}^{*}$ is not weak*-closed.

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