WEAK*-CLOSURE OF CERTAIN SUBSPACES OF THE DUAL OF SOME ABELIAN BANACH ALGEBRAS

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ABSTRACT. We shall determine the weak*-closure of some subspaces of the dual of certain Banach algebras. In particular, we consider this problematic in the context of abelian C*-algebras.

1. INTRODUCTION

Let \( I \) be a closed ideal of an abelian Banach algebra \( \mathcal{A} \) with bounded approximate identity. The determination of conditions under which \( I \) itself is provided with a bounded approximate identity is a matter of interest. Recently, it was shown how certain idempotents of the second dual space \( \mathcal{A}^{**} \) of \( \mathcal{A} \) are related with this problem. In particular, \( \mathcal{A}^{**} \) is considered with the Banach algebra structure given by any one of the two canonical Arens products, in such a way that \( \mathcal{A}^{**} \) becomes a Banach subalgebra of \( \mathcal{A}^{**} \) by means of the natural isometric immersion \( \chi_A : \mathcal{A} \to \mathcal{A}^{**} \). In this framework, \( I \) has a bounded approximate identity if and only if there is an idempotent \( a^{**} \in \mathcal{A}^{**} \) so that the space \( a^{**} \mathcal{A}^* \) is weak*-closed in \( \mathcal{A}^* \) and \( I = \{ a \in \mathcal{A} : aa^{**} = 0 \} \) (cf. [8, Lemma 2.3]). Consequently, the characterization of elements \( a^{**} \in \mathcal{A}^{**} \) so that \( a^{**} \mathcal{A}^* \) is weak*-closed is an issue that deserves consideration and probably is interesting on its own.

2. ABOUT THE WEAK*-CLOSEDNESS OF \( a^{**} \mathcal{A}^* \)

Theorem 1. Given \( a^{**} \in \mathcal{A}^{**} \) let \( \rho_{a^{**}} : \mathcal{A} \to \mathcal{A}^{**} \) be the map \( \rho_{a^{**}}(a) = aa^{**} \) if \( a \in \mathcal{A} \).

1. The following equalities hold:

\[
(a^{**} \mathcal{A}^*)^{w^*} = \bigcap \{ \ker \chi_{a^{**}}(a) : a \in \mathcal{A} \in \mathcal{A} \}
= \bigcap \{ \ker \chi_{a^{**}}(a) : aa^{**} = 0 \}
= (\ker(\rho_{a^{**}}))^{w^*}.
\]

2. Let \( \mathcal{A} \) be a weakly compact Banach algebra. Then \( a^{**} \mathcal{A}^* \) is weak*-closed in \( \mathcal{A}^* \) if and only if \( \mathcal{R}[(\chi_{a^{**}})^{-1} \circ \rho_{a^{**}}] \) is closed in \( \mathcal{A} \).

Proof. (1) As the weak*-topology is locally convex by the Hahn–Banach separation theorem is \( (a^{**} \mathcal{A}^*)^{w^*} = \bigcap_{\gamma \in \Gamma} \ker(\gamma) \), where \( \Gamma \) denotes the set of weak*-continuous linear forms that annihilates on \( a^{**} \mathcal{A}^* \) (cf. [5, Cor. 1.2.13]). Moreover, any weak*-continuous linear form \( \gamma \) is realized as an evaluation, i.e., \( \gamma \in \mathcal{S}(\mathcal{A}^{**}) \) (cf. [6, Prop. 1.3.5]). The second and third equalities are immediate.

(2) Since \( \mathcal{A} \) is weakly compact \( \chi_{a^{**}}(\mathcal{A}) \) becomes a two sided ideal of \( \mathcal{A}^{**} \). So \( (\chi_{a^{**}})^{-1} \circ \rho_{a^{**}} \in \mathcal{B}(\mathcal{A}) \) and \( a^{**} \mathcal{A}^* = ((\chi_{a^{**}})^{-1} \circ \rho_{a^{**}})(\mathcal{A}) \) for all \( x^* \in \mathcal{A}^* \). Now the assertion follows by ([4, Ch. VI, Th. 1.10]).

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Corollary 1. If $a^{**} \in \mathcal{A}^{**}$, $a^{**} \mathcal{A}^{*}$ is weak*-closed if and only if $(\ker(\rho_{a^{**}}))^{\perp} \subseteq a^{**} \mathcal{A}^{*}$.

Example 1. Let $\mathcal{A} = c_0(S)$ be the usual Banach algebra of continuous functions vanishing at infinity on an infinite discrete set $S$. Then $\mathcal{A}^{**} \approx l^1(S)$ and $\mathcal{A}^{*} \approx l^\infty(S)$. Given $a^{**} \in \mathcal{A}^{**}$ let us write $\sigma(a^{**}) = \{ s \in S : a^s \neq 0 \}$. For $s \in S$ let $\delta^s \in \mathcal{A}^{*}$ be the current delta function on $S$ so that $\delta^s(s) = 1$ and $\delta^s(t) = 0$ if $t \in S - \{ s \}$. By Theorem 1 and as $\{ \delta^s \}_{s \in S}$ is a basis of $\mathcal{A}^{*}$ it is easy to see that

$$\mathcal{A}^{**} = \operatorname{span}\{ s \in \sigma(a^{**}) \} = \mathcal{A}^{*}.$$

So $a^{**} \mathcal{A}^{*}$ is weak*-closed if and only if $\sum_{s \in \sigma(a^{**})} |a^s/a^{**}| < \infty$ whenever $\sum_{s \in \sigma(a^{**})} |a^s| < \infty$. This condition holds if $\sigma(a^{**})$ is finite. Otherwise, let $P_f(\sigma(a^{**}))$ be the class of finite subsets of $\sigma(a^{**})$ and if $F \in P_f(\sigma(a^{**}))$ let $T_F \in l^1(\sigma(a^{**}))$ so that $T_F(a^s) = \sum_{s \in F} a^s/a^{**}$. By the uniform boundedness principle the class $\{ T_F \}_{F \in P_f(\sigma(a^{**}))}$ becomes bounded. But $\| T_F \| = \max \{ |a^s| : s \in F \}$ for each $F$. Consequently, $a^{**} \mathcal{A}^{*}$ is weak*-closed if and only if $\inf \sigma(a^{**}) > 0$.

3. WHEN $\mathcal{A}$ IS AN ABELIAN $C^*$-ALGEBRA

Theorem 2. Let $\mathcal{A}$ be an abelian non-reflexive $C^*$-algebra and let $a^{**} \in \mathcal{A}^{**}$.

1. If $a^{**}$ is invertible or quasi-nilpotent then $a^{**} \mathcal{A}^{*}$ is weak*-closed.

2. Let $a^{**} \in \mathcal{A}^{**} - \chi_{\mathcal{A}}(\mathcal{A}^{*})$ be idempotent.
   (a) There exists a weak*-dense norm-closed subspace $\Sigma_{a^{**}}$ of $\ker(\rho_{a^{**}})^{\perp}$ so that $a^{**} \mathcal{A}^{*} \subseteq \chi_{\mathcal{A}}(\mathcal{A}^{*}) \oplus \Sigma_{a^{**}}^{\perp}$.

(b) Let $\{ a_i \}_{i \in T}$ be a bounded net of $\mathcal{A}$ so that $a^{**} = w^* - \lim_{i \in T} \chi_{\mathcal{A}}(a_i)$. The set $\mathcal{J} = \{ a \in \mathcal{A} : \exists x \in \mathcal{A} / a = w - \lim_{i \in T}(x a_i) \}$ is an ideal of $\mathcal{A}$. Further, given $a \in \mathcal{A}$, $\chi_{\mathcal{A}}(a) \in \mathcal{A}^{**}$ if and only if $a \in \mathcal{J}$.

(c) $a^{**} \mathcal{A}^{*}$ is weak*-closed if and only if $a^{**} \mathcal{A}^{*} = \mathcal{A}^{*}$.

Proof. Since $\mathcal{A}$ is a complex abelian $C^*$-algebra it becomes Arens regular [10]. Indeed, the second conjugate algebra $(\mathcal{A}^{**}, \square)$ becomes a $C^*$-algebra that is abelian because $\mathcal{A}$ is and unital (cf. [12 Th 7.1]). Besides $\mathcal{A}$ has a bounded approximate identity and so $(\mathcal{A}^{**}, \square)$ is unital. If $\Delta(\mathcal{A}^{**})$ denotes the maximal ideal space of $(\mathcal{A}^{**}, \square)$ the Gelfand transform $G : (\mathcal{A}^{**}, \square) \to C(\Delta(\mathcal{A}^{**}))$ provides an isometric isomorphism of Banach algebras.

1. Given $a^{**} \in \mathcal{A}^{**}$ let $a \in \ker(\rho_{a^{**}})$. Then $0 \in \ker(\rho_{a^{**}})G(\chi_{\mathcal{A}}(a)) = G(\chi_{\mathcal{A}}(a))G(a^{**})$.
   i.e., $h(\chi_{\mathcal{A}}(a))h(a^{**}) = 0$ for all $h \in \Delta(\mathcal{A}^{**})$.
   If $a^{**}$ is invertible it is clear that $\sigma_{a^{**}}(\chi_{\mathcal{A}}(a)) = \{ 0 \}$ and $\sigma_{a^{**}}(a) = \{ 0 \}$ because $\chi_{\mathcal{A}}$ is isometric. So it is readily seen that $a = 0_{\mathcal{A}}$ and $(\ker(\rho_{a^{**}}))^{\perp} = a^{**}$.
   If $a^{**}$ is quasi-nilpotent $\ker(\rho_{a^{**}}) = \mathcal{A}$ and $(\ker(\rho_{a^{**}}))^{\perp} = \{ 0_{\mathcal{A}} \}$.
   In both cases it is plain that $a^{**} \mathcal{A}^{*}$ becomes weak*-closed if $a^{**} \in \mathcal{A}^{**}$ is invertible or quasi-nilpotent and the claim follows.

2. Let us write $I(a^{**}) = G(a^{**})^{-1}[\sigma_{a^{**}}(a^{**}) - \{ 0 \}]$. We have $\ker(\rho_{a^{**}}) = \cap \{ \ker(h \circ \chi_{\mathcal{A}}) : h \in I(a^{**}) \}$.

Therefore

$$\ker(\rho_{a^{**}}) = \operatorname{span}[\cup_{h \in I(a^{**})} \ker(h \circ \chi_{\mathcal{A}})^{\perp}] = \sigma_{a^{**}}^{\perp}.$$

Actas del XIV Congresso Dr. Antonio A. R. Monteiro (2017), 2019
with $\Sigma_{a^{**}} = \text{span}\{S_{a^{**}}\}$ and $S_{a^{**}} = \{h \circ \chi_{a^{**}} : h \in I(a^{**})\}$. If $h \in \Delta(a^{**})$ it is straightforward to see that $a^{**}(h \circ \chi_{a^{**}}) = (h \circ \chi_{a^{**}}, a^{**})h \circ \chi_{a^{**}}$. Further, let $h \in I(a^{**})$ so that $\langle h \circ \chi_{a^{**}}, a^{**} \rangle \neq 0$. If $a^{**}, a^{**}$ is weak*-closed we can write $h \circ \chi_{a^{**}} = a^{**}x_{a}^{*}$ for some $x_{a}^{*} \in a^{**}$.

(a) If $a^{**}$ is idempotent we have

$$a^{**}x_{a}^{*} = a^{**}(h \circ \chi_{a^{**}}) = \langle h \circ \chi_{a^{**}}, a^{**} \rangle h \circ \chi_{a^{**}} = h \circ \chi_{a^{**}}.$$ 

Then $\langle h \circ \chi_{a^{**}}, a^{**} \rangle = 1$ and $a^{**}(h \circ \chi_{a^{**}}) = h \circ \chi_{a^{**}}$. Thus, if $a \in a^{**}$ and $a^{*} \in \Sigma_{a^{**}}$ we see that

$$\langle a^{*}, \chi_{a^{**}}(a) \rangle = \langle a, a^{*} \rangle = \langle a, a^{**}a^{*} \rangle = \langle a^{*}a, a^{*} \rangle = \langle a^{*}, \rho_{a^{**}}(a) \rangle,$$

i.e., $\rho_{a^{**}}(a) - \chi_{a^{**}}(a) \in [\Sigma_{a^{**}}]^{\perp}$. If $\chi_{a^{**}}(a) \in [\Sigma_{a^{**}}]^{\perp}$ let us consider a nonzero homomorphism $\phi_{0} : a^{**} \to \mathbb{C}$. It admits a natural extension, on the $C^{*}$-subalgebra $\mathcal{A}$ of $a^{**}$ generated by $\chi_{a^{**}}(a^{**}) \cup \{a^{**}\}$, to an isomorphism $\phi_{1}$ such that $\phi_{1}(a^{**}) = 1$. Since $\mathcal{A}$ is a commutative symmetric Banach *-algebra its Shilov boundary $\partial \mathcal{A}$ coincides with the whole maximal ideal space $\Delta(\mathcal{A})$ (cf. [7] Example 3.3.16)). Thus $\phi_{1}$ has an extension to a character $\phi_{2} \in a^{**}$ (cf. [4] Cor. 1]). Hence $\phi_{2} \in I(a^{**})$, $\phi_{2} \circ \chi_{a^{**}} \in \Sigma_{a^{**}}$ and

$$0 = \langle \phi_{2} \circ \chi_{a^{**}}, \chi_{a^{**}}(a) \rangle = \langle a, \phi_{2} \circ \chi_{a^{**}} \rangle = \langle \chi_{a^{**}}(a), \phi_{2} \rangle = \langle \phi_{2}, \phi_{0} \rangle.$$ 

Thus $a \in a^{**}$ must be quasi-nilpotent and we can conclude that $a = 0_{a^{**}}$, i.e., $a^{**} \subseteq \chi_{a^{**}}(a^{**}) [\Sigma_{a^{**}}]^{\perp}$.

(b) It is easy to see that $\mathcal{J}$ is an ideal of $a^{**}$, eventually trivial. Let $a \in a^{**}$ so that $\chi_{a^{**}}(a) = xa^{**}$ for some $x \in a^{**}$. Given $a^{*} \in a^{**}$ we have

$$\langle a^{*}, a^{*} \rangle = \langle a^{*}x, a^{**} \rangle = \lim_{t \in \mathcal{J}} \langle a_{t}, a^{*}x \rangle = \langle xa_{t}, a^{*} \rangle,$$

i.e., $a \in \mathcal{J}$. Likewise, if $a \in \mathcal{J}$ and $a = w - \lim_{t \in \mathcal{T}} (xa_{t})$ for some $x \in a^{**}$ then

$$\langle a^{*}, xa^{**} \rangle = \lim_{t \in \mathcal{T}} \langle a_{t}, a^{*}x \rangle = \langle a, a^{*} \rangle$$

and so $\chi_{a^{**}}(a) = xa^{**}$.

(c) Since $(\ker(\rho_{a^{**}}))^{\perp} = [\chi_{a^{**}}(\ker(\rho_{a^{**}}))]^{\perp}$, by (1) we see that

$$\chi_{a^{**}}(\ker(\rho_{a^{**}})) \subseteq [\Sigma_{a^{**}}]^{\perp}.$$ 

Consequently $\rho_{a^{**}}$ must be injective and the assertion follows by Corollary [1].

\[\square\]

**Example 2.** Let $a^{**} = C_{0}(G)$ be the uniform Banach algebra of complex functions vanishing at infinity on a locally compact abelian group $G$ with Haar invariant measure $\lambda$ and identity element $e$. There is an isometric isomorphism of Banach spaces between $a^{**}$ and the Banach space $M(G)$ of complex bounded regular Borel measures (cf. [9] Th. 2.14)). Further, by the Lebesgue–Radon–Nikodym decomposition theorem,

$$a^{**} \cong l^{1}(G) \oplus l^{1}(G, \lambda) \oplus M_{c}(G, \lambda).$$
i.e., any element of $M(G)$ can be represented uniquely as the sum of a discrete measure, an absolutely continuous and a singular continuous measure with respect to $\lambda$ (cf. [9, Th. 6.10]). Consequently,

$$\mathcal{A}^{**} \approx L^\infty(G) \oplus \mu L^\infty(G) \oplus \mu \mathcal{M}_c(G, \lambda)^*.$$ 

Let $m, n \in L^\infty(G)$, $M, N \in L^\infty(G)$ and $\mu, \eta \in \mathcal{M}_c(G, \lambda)^*$. It is straightforward to see that

$$(m, n, \mu(n, N, \eta) = (mN, \mu \square \eta),$$

where $\square$ is the current Arens product of $\mathcal{A}^{**}$ and $mn$ and $MN$ represent the pointwise products of $L^\infty(G)$ and $L^\infty(G)$. Let $W$ be a compact neighbourhood of $e$ and let $M = I_W - \{e\}$, where $I_W$ is the usual characteristic function of $W - \{e\}$. Then $M$ can be viewed as an idempotent of $\mathcal{A}^{**}$ and $M\mathcal{A}^*$ is not weak*-closed. For instance, let $\mathcal{N} = \{\lambda(U)^{-1}I_U\}_{U \in \mathcal{U}_e}$, where $\mathcal{U}_e$ is the directed set of relatively compact symmetric neighbourhoods of $e$. The net $\mathcal{N}$ is bounded in $L^1(G)$ and $\delta_\varepsilon = \omega^\varepsilon - \lim_{U \in \mathcal{U}_e} F[\lambda(U)^{-1}I_U]$, i.e., $\delta_\varepsilon \in (M\mathcal{A}^*)^{-w^*}$. Nevertheless, $M\mathcal{A}^* = ML^1(G) \oplus ML^1(G, \lambda) \oplus 1(0_{\mathcal{M}_c(G, \lambda)^*})$ and

$$ML^1(G) = \{\zeta \in L^1(G) : \text{supp}(\zeta) \subseteq W - \{e\}\}$$

and therefore $M\mathcal{A}^*$ is not weak*-closed.

References


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