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CONVERGENCE RESULTS FOR DIRICHLET SERIES ON THE LINE $1+it$

PABLO A. PANZONE

ABSTRACT. D. J. Newman gave a new proof of a convergence result for bounded coefficient Dirichlet series (due to A. E. Ingham) which leads to a simple proof of the prime number theorem. In this paper we prove a generalization of the Ingham–Newman theorem.

1. INTRODUCTION AND RESULTS

D. J. Newman [4] gave a simple and surprising proof, using complex variables and contour integration techniques, of the following theorem.

Theorem 1.1. *Suppose $|a_n| \leq 1$ and form the series $\sum_{n=1}^{\infty} a_n n^{-z}$ which clearly converges to an analytic function $F(z)$ for $\Re z > 1$. Assume $F(z)$ is analytic throughout $\Re z \geq 1$. Then $\sum_{n=1}^{\infty} a_n n^{-z}$ converges to $F(z)$ throughout $\Re z \geq 1$.*

The theorem can be used to give a a simple analytic proof of the prime number theorem; for details see [4] and for a more recent proof of the prime number theorem see [7]. The above theorem is a very special case of a theorem of A. E. Ingham (proved 47 years earlier, see [3, Theorem 3 (1), p. 461]), whose investigations and results are much broader and richer.

If one writes

$$S_N(z) := \sum_{n=1}^{N} \frac{a_n}{n^z},$$

$$r_N(z) := F(z) - S_N(z) = \sum_{n=N+1}^{\infty} \frac{a_n}{n^z},$$

then Theorem 1.1 states, in other words, that for any real t, $S_N(1+it) \to F(1+it)$, or equivalently $r_N(1+it) \to 0$, as $N \to \infty$. A natural question to ask is what happens with the derivatives of these functions.

Question *Under the conditions of Theorem 1.1, is it true that*

$$S_N'(z) \to F'(z) \quad (\Re z = 1,\, N \to \infty)?$$

In this paper we provide some partial answers. For example we prove that:

- *Under the conditions of Theorem 1.1 one has*

$$S_N'(z) + \frac{S_N''(z)}{\log N} \to F'(z) \quad (\Re z = 1,\, N \to \infty).$$

- *Under the conditions of Theorem 1.1 and a certain mild growth condition on F one has*

$$S_N'(z) \to F'(z) \quad (\Re z = 1,\, N \to \infty).$$

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The method of proof is an extension of Newman's proof.

After this introduction we state explicity our results. Denote by W_α the set of complex numbers of the form $w = 1+it$ with t real and belonging to $[-\alpha,\alpha]$, $0<\alpha$.

Theorem 1.2. *Under the hypothesis of Theorem 1.1 for any non-negative integer n*

$$\sum_{i=0}^{n}\binom{n}{i}\frac{S_N^{(n+i)}(w)}{\log^i N}\to F^{(n)}(w), \quad \text{if } N\to\infty. \tag{1.1}$$

Also for any $n\geq 1$

$$\sum_{i=1}^{n}\binom{n}{i}(-1)^{i+1}\frac{i}{\log^i N}\sum_{j=1}^{N-1}S_j^{(n)}(w)\frac{\log^{i-1} j}{j}\to F^{(n)}(w), \quad \text{if } N\to\infty. \tag{1.2}$$

Moreover, the convergence is uniform in W_α *for any* $\alpha>0$.

Definition 1.3. We say that F satisfies the growth condition $\mathrm{GC}(\lambda)$ if there exist constants $r>1$, $A_1>0$ such that for any $R>R_0>0$ one has

$$F(x+iy)=O(R^\lambda),$$

whenever $|y|\leq R$, $1-A_1e^{-\frac{\log R}{\log_r R}}\leq x<1$, assuming that F is analytic in such region (here we write $\log_r R=\log(\cdots\log R)$ r times, with r a natural parameter).

We say that F satisfies the growth condition $\mathrm{GC}(\lambda,\beta)$ if one has the same bound for F but in the region $|y|\leq R$, $1-\frac{A_1}{R^\beta}\leq x<1$, for some positive β, where we assume that F is analytic in such region.

We will use the above definition with $\lambda=3/2$. Observe that this compares favorably with known results for the Riemann zeta function, i.e. taking $F(z)=1/\zeta(z)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^z}$ it is known, for example, that

$$F(x+iy)=O(\log^7 R),$$

whenever $|y|\leq R$, $1-A_1e^{-9\log(\log R)}\leq x<1$ and $R>e$ ([1, p. 291, Theorem 13.10]).

Observe that the regions in the definition of any of our growth conditions are, roughly speaking, thinner than the region given for $1/\zeta(z)$. We prove the following theorem.

Theorem 1.4. *Under the hypothesis of Theorem 1.1 the following holds.*

a) *If F satisfies the growth condition GC(3/2), then for any* $m=0,1,2,\ldots$

$$S_N^{(m)}(1)\to F^{(m)}(1) \quad \textit{and} \quad r_N(1)\log^m N\to 0, \quad \text{if } N\to\infty. \tag{1.3}$$

b) *Let* β,λ *be positive real numbers and* m_0 *a positive integer such that* $m_0\beta<1$ *and* $m_0(1-\lambda)+2>0$. *If F satisfies the growth condition* $GC(\lambda,\beta)$ *then* (1.3) *is true for* $m=1,2,3,\ldots,m_0$.

Corollary 1.5. *Assume the hypothesis of Theorem 1.1 and that F satisfies the growth condition GC(*λ*) for some* $\lambda>0$. *Furthermore assume that for some positive constants* $c<1$, *A one has*

$$|F(x+iy)|=Ae^{e^{c|y|}},$$

if $1<x$. *Then for any* $m=0,1,2,\ldots$

$$S_N^{(m)}(1)\to F^{(m)}(1) \quad \textit{and} \quad r_N(1)\log^m N\to 0, \quad \text{if } N\to\infty.$$

In case that $F(z) = 1/\zeta(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$ the above theorems have been known for some time. Their proofs used free zero regions of the zeta function [2, 5, 6].

At this point it is fair to say that we were unable to find a function for which Theorem 1.4 applies but for which classical methods do not. In particular we do not know if there exist functions $F(z)$ satisfying the hypothesis of Theorem 1.1 such that $F(z) = \Omega(R^{\beta})$, for some $\beta, A_1 > 0$, in the region $|y| \leq R$, $1 - A_1 e^{-\frac{\log R}{\log_r R}} \leq x < 1$.

Our main results, that is Theorem 1.2, Theorem 1.4 and Corollary 1.5, will follow from the following key lemma which is proved in the next section.

Lemma 1.6. *Assume the hypothesis of Theorem 1.1. Let $\ell, k = 1, 2, 3, \ldots$ with $\ell \geq k$. Let γ be a small circle around zero in which $F(z+w)$ is defined as a function of z where w is such that $\Re w = 1$. Set*

$$I_w = I_{w,k,\ell,R,N} := \int_{\gamma} \big(F(z+w) - S_N(z+w)\big)^k \Big(\frac{1}{z} + \frac{z}{R^2}\Big)^{\ell} N^{kz}\, dz.$$

a) Fix an $\alpha > 0$. Then for any $R > 2\alpha > 0$ there exist $\delta > 0$ and a constant $C_0 = C_0(\delta, R, \alpha, \ell, k)$, such that for any natural N and any $w \in W_{\alpha}$ one has

$$|I_w| \leq \frac{\pi 2^{\ell+k+1}}{R^{\ell+k-1}} + C_0\Big(\frac{1}{\log^2 N} + \frac{1}{N^{\delta}}\Big).$$

b) If F also satisfies the growth condition GC(3/2), then for any integer $m > 0$ one has

$$I_{1,k,\ell,\log^{2m} N,N} \log^m N \to 0, \quad \text{if } N \to \infty.$$

c) Let β, λ be positive real numbers and m_0 a positive integer such that $m_0\beta < 1$ and $m_0(1-\lambda) + 2 > 0$. If F satisfies the growth condition $GC(\lambda, \beta)$ then there exists $\varepsilon > 0$ such that

$$I_{1,1,\ell,\log^{m_0+\varepsilon} N,N} \log^{m_0} N \to 0, \quad \text{if } N \to \infty.$$

To prove the theorems we need the following two easy lemmas.

Lemma 1.7. *Under the hypothesis of Theorem 1.1 if $N \to \infty$ one has*

i) $r_N(w) \to 0$,
ii) $r_N(w)\log N + r_N(w)' \to 0$,
iii) $r_N(w)\log^2 N + 2r_N'(w)\log N + r_N''(w) \to 0$, *and in general if $n = 0, 1, 2, 3, \ldots$*

$$L_n(w) := \sum_{j=0}^{n} \log^j N \binom{n}{j} r_N^{(n-j)}(w) \to 0.$$

Moreover, the convergence is uniform if $w \in W_{\alpha}$ for any $\alpha > 0$.

Proof. We calculate the integral I_w of Lemma 1.6 in two ways.

Firstly, by Lemma 1.6 part (a) and for fixed k, ℓ, α, the integral I_w tends uniformly to zero if $w \in W_{\alpha}$ by taking first R large and then N large enough.

Secondly, the integral I_w can be calculated with Cauchy integral formulae. The case $k = \ell = 1$ is inmmediate (this is Theorem 1.1 without the requirement of uniformity). This gives case (i).

Taking $k = 1$, $\ell = 2$ the integral I_w is

$$2\pi i \big\{ r_N(w)\log N + r_N'(w) \big\},$$

which gives case (ii).

If $k=1$, $\ell=3$ the integral I_w is

$$2\pi i\left\{\frac{3}{R^2}r_N(w)+\frac{r_N''(w)}{2}+r_N'(w)\log N+\frac{r_N(w)\log^2 N}{2}\right\},$$

and this gives case (iii), noticing that we have already proved (i).

The general case is obtained using $k=1$, $\ell=n+1$, Leibnitz rule for derivatives and induction. □

Lemma 1.8. *For any natural number n and $i=1,2,\dots,n$ one has*

$$\frac{S_N^{(n+i)}(w)}{\log^i N}=(-1)^i\left\{S_N^{(n)}(w)-\frac{i}{\log^i N}\sum_{j=1}^{N-1}S_j^{(n)}(w)\frac{\log^{i-1} j}{j}\right\}+o(1),$$

if $N\to\infty$, where the $o(1)$ term tends uniformly to zero on the line $w=1+it$, $t\in R$.

Proof. Recall Abel's summation formula: if $D_j=\sum_{k=1}^j d_k$ then

$$\sum_{j=1}^N b_j d_j=b_N D_N+\sum_{j=1}^{N-1}(b_j-b_{j+1})D_j.$$

But putting $d_j=\frac{a_j}{j^w}\log^n j$ and $b_j=\log^i j$ gives

$$\begin{aligned}(-1)^{n+i}S_N^{(n+i)}(w)&=\sum_{j=1}^N\frac{a_j}{j^w}\log^{n+i} j\\&=(-1)^n\left\{\log^i N S_N^{(n)}(w)+\sum_{j=1}^{N-1}S_j^{(n)}(w)(\log^i j-\log^i(j+1))\right\}.\end{aligned}$$

Dividing this equality by $(-1)^{n+i}\log^i N$ gives

$$\frac{S_N^{(n+i)}(w)}{\log^i N}=(-1)^i\left\{S_N^{(n)}(w)+\frac{1}{\log^i N}\sum_{j=1}^{N-1}S_j^{(n)}(w)(\log^i j-\log^i(j+1))\right\}.$$

Notice that for fixed i, as j tends to infinity,

$$\log^i j-\log^i(j+1)=-\frac{i}{j}\log^{i-1} j+O\Big(\frac{\log^{i-1} j}{j^2}\Big).$$

(Hint: use the identity $\beta^i-\theta^i=(\beta-\theta)(\beta^{i-1}+\beta^{i-2}\theta+\cdots+\theta^{i-1})$ and the fact that $\log j-\log(j+1)=-\frac{1}{j}+O(\frac{1}{j^2})$ as j tends to infinity.) The result follows inserting this equation into the above equation and noting that as $S_j^{(n)}(w)=O(\log^{n+1} j)$ one has

$$O\Big(\sum_{j=1}^{N-1}S_j^{(n)}(w)\frac{\log^{i-1} j}{j^2}\Big)=O\Big(\sum_{j=1}^{N-1}\frac{\log^{n+i} j}{j^2}\Big)=O(1).$$ □

Proof of Theorem 1.2. We first deal with (1.1) and $n=1$: dividing equation (iii) of Lemma 1.7 by $\log N$ and substracting equation (ii) gives

$$r_N'(w)+\frac{r_N''(w)}{\log N}\to 0\quad\text{if } N\to\infty,$$

which can be rewritten as

$$S_N'(w)+\frac{S_N''(w)}{\log N}\to F'(w)\quad\text{if } N\to\infty,$$

the convergence being uniform if $w\in W_\alpha$. This gives formula (1.1) with $n=1$.

Inserting the identity of Lemma 1.8 with $n = i = 1$ into this last equation gives formula (1.2) with $n = 1$.

The general case, that is, formula (1.1), follows the same ideas and it is as follows. Firstly, fix n and take the following linear combination of terms $L_j(w)$, as defined in Lemma 1.7,

$$(-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{L_{n+i}(w)}{\log^i N} = \sum_{i=0}^{n} \binom{n}{i} \frac{r_N^{(n+i)}(w)}{\log^i N},$$

where the equality follows using the properties of the binomial coefficients. Using Lemma 1.7 and the fact that $F^{(j)}(w)/\log^i N \to 0$ for any $j, i > 0$ as $N \to \infty$ the above simplifies to

$$\sum_{i=0}^{n} \binom{n}{i} \frac{S_N^{(n+i)}(w)}{\log^i N} \to F^{(n)}(w),$$

uniformly in $w \in W_\alpha$ if $N \to \infty$, which yields (1.1). Inserting the identities of Lemma 1.8 into this last equation gives

$$S_N^{(n)}(w) + \sum_{i=1}^{n} \binom{n}{i} (-1)^i \left\{ S_N^{(n)}(w) - \frac{i}{\log^i N} \sum_{j=1}^{N-1} S_j^{(n)}(w) \frac{\log^{i-1} j}{j} \right\} \to F^{(n)}(w)$$

as $N \to \infty$, which gives formula (1.2) after simplification. □

Proof of Theorem 1.4. a) Take $k = 1$, $\ell = 1$ in Lemma 1.6 (b). Then this gives (using (i) of Lemma 1.7) that for any integer $m \geq 0$

$$r_N(1) \log^m N \to 0$$

as $N \to \infty$. Next take $k = 1, \ell = 2$. Using (ii) of Lemma 1.7 gives

$$\left\{ r_N'(1) + r_N(1) \log N \right\} \log^m N \to 0$$

as $N \to \infty$ and therefore

$$r_N'(1) \log^m N \to 0,$$

as $N \to \infty$.

Next we take $k = 1$, $\ell = 3$ and the same ideas apply. This proves part (a) of Theorem 1.4.

b) The proof is similar but now one has

$$r_N(1) \log^{m_0} N \to 0, \quad \left\{ r_N'(1) + r_N(1) \log N \right\} \log^{m_0 - 1} N \to 0, \quad \ldots$$

as $N \to \infty$. Therefore

$$r_N^{(m_0)}(1) \to 0, \quad r_N^{(m_0-1)}(1) \to 0, \quad \ldots$$

as $N \to \infty$. This ends our proof. □

2. Proof of Lemma 1.6

Proof. We define the curves A, B, C, D as follows. The curve A is $|z| = R > 1$, $\Re z \geq 0$. The curve B is $|z| = R$, $-\delta \leq \Re z \leq 0$ (strictly speaking, these are two curves). The curve C is $|z| = R$, $\Re z \leq 0$. The curve D is $|z| \leq R$, $\Re z = -\delta$. By the hypothesis, given a number $R > 2\alpha > 0$ there exists a number $\delta > 0$ such that $F(z+w)$ is analytic on and inside the curve $A + B + D$ for any $w = 1 + it$, t belonging to a fixed real interval $[-\alpha, \alpha]$, that is, $w \in W_\alpha$. Moreover, by a standard argument, assume that M_0 is the supremum of $|F(z+w)|$ on $B + D$ for such w. Note that M_0 depends on δ, R, α. If F satisfies the growth condition GC(3/2) then M_0, the supremum of $|F(z+1)|$ on $B + D$, is $M_0 = O(R^{3/2})$, and $\delta = A_1 e^{-\frac{\log R}{\log_r R}}$.

Write for short

$$G(z) := \left(\frac{1}{z} + \frac{z}{R^2} \right)^{\ell} N^{kz}.$$

Then by Cauchy's theorem

$$\begin{aligned}
I_w &= \int_{A+B+D} \Big(F(z+w) - S_N(z+w)\Big)^k G(z)\,dz \\
&= \int_A r_N(z+w)^k G(z)\,dz + \int_{B+D} \Big(F(z+w) - S_N(z+w)\Big)^k G(z)\,dz \\
&= \int_A r_N(z+w)^k G(z)\,dz + J.
\end{aligned}$$

Expanding $\Big(F(z+w)-S_N(z+w)\Big)^k$ using the binomial theorem in this last integral, one sees that to estimate J it is enough to estimate the integrals

$$J_i = \int_{B+D} F(z+w)^i S_N(z+w)^{k-i} G(z)\,dz,$$

with $i = 1,2,\ldots,k$ and

$$J_0 = \int_{B+D} S_N(z+w)^k G(z)\,dz = \int_C S_N(z+w)^k G(z)\,dz$$

$$= (-1)^{\ell+1} \int_A S_N(-z+w)^k \Big(\frac{1}{z} + \frac{z}{R^2}\Big)^{\ell} N^{-kz}\,dz.$$

In the last formula, the second equality again follows from Cauchy's theorem and the last equality follows changing variables $z \to -z$. Observe that

$$J = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} J_i. \tag{2.1}$$

Recall the following estimates from [4]:

$$\Big(\frac{1}{z} + \frac{z}{R^2}\Big) = \frac{2x}{R^2}, \quad \text{if } |z| = R. \tag{2.2}$$

$$\Big|\frac{1}{z} + \frac{z}{R^2}\Big| \le \frac{1}{\delta}\Big(1 + \frac{|z|^2}{R^2}\Big) \le \frac{2}{\delta}, \qquad \text{if } \Re z = x = -\delta \text{ and } |z| \le R. \tag{2.3}$$

$$\Big|r_N(z+w)\Big| \le \sum_{n=N+1}^{\infty} \frac{1}{n^{x+1}} \le \int_N^{\infty} \frac{dn}{n^{1+x}} \le \frac{1}{xN^x}, \quad \text{if } x > 0. \tag{2.4}$$

$$\Big|S_N(w-z)\Big| \le \sum_{n=1}^{N} \frac{1}{n^{1-x}} \le N^{x-1} + \int_0^N n^{x-1}\,dn \le N^x\Big(\frac{1}{N} + \frac{1}{x}\Big), \quad \text{if } x > 0. \tag{2.5}$$

The proof of the lemma now goes as follows.

Using the usual maximum-times-length estimation for integrals of complex variable one has firstly (recall $\ell \ge k$):

$$\Big|\int_A r_N(z+w)^k G(z)\,dz\Big| \le \max_{0\le x\le R} \Big(\frac{1}{xN^x}\Big)^k \Big(\frac{2x}{R^2}\Big)^{\ell} N^{kx} \pi R \le \frac{\pi 2^{\ell}}{R^{\ell+k-1}}.$$

Using formulas (2.2), (2.5) the estimate for J_0 is

$$\begin{aligned}\Big|J_0\Big| &= \Big|\int_A S_N(-z+w)^k\Big(\frac{1}{z}+\frac{z}{R^2}\Big)^\ell N^{-kz}\,dz\Big| \le \max_{0\le x\le R} N^{kx}\Big(\frac{1}{N}+\frac{1}{x}\Big)^k\Big(\frac{2x}{R^2}\Big)^\ell N^{-kx}\pi R\\
&\le \frac{\pi 2^\ell}{R^{2\ell-1}}\max_{0\le x\le R}\Big(\frac{1}{N}+\frac{1}{x}\Big)^k x^\ell \le \frac{\pi 2^{\ell+k}}{R^{2\ell-1}}\max_{i=0,\dots,k}\max_{0\le x\le R}\frac{1}{N^{k-i}x^i}x^\ell\\
&= \frac{\pi 2^{\ell+k}}{R^{2\ell-1}}\max_{i=0,\dots,k}\frac{R^{\ell-i}}{N^{k-i}} \le \frac{\pi 2^{\ell+k}}{R^{2\ell-1}}\Big\{\frac{R^\ell}{N^k}+\frac{R^{\ell-1}}{N^{k-1}}+\cdots+R^{\ell-k}\Big\}\\
&\le \frac{\pi 2^{\ell+k}k}{N}+\frac{\pi 2^{\ell+k}}{R^{\ell+k-1}}\end{aligned}$$

(in the last inequality recall that $R>1$). Observe that using formula (2.1) and these last two inequalites one gets

$$|I_w| \le \frac{\pi 2^{\ell+k+1}}{R^{\ell+k-1}}+\frac{\pi 2^{\ell+k}k}{N}+2^k\max_{i=1,\dots,k}|J_i|. \tag{2.6}$$

Next we estimate $|J_i|$. Recall that $J_i=\int_{B+D}F(z+w)^i S_N(z+w)^{k-i}G(z)\,dz$. One has

$$\Big|J_i\Big| \le \Big|\int_B\Big|+\Big|\int_D\Big|,$$

and using formulas (2.3), (2.5),

$$\begin{aligned}\Big|\int_D\Big| &\le M_0^i\Big\{N^\delta\Big(\frac{1}{N}+\frac{1}{\delta}\Big)\Big\}^{k-i}\Big(\frac{2}{\delta}\Big)^\ell N^{-k\delta}2R\\
&\le M_0^i\Big(1+\frac{1}{\delta}\Big)^{k-i}\Big(\frac{2}{\delta}\Big)^\ell N^{-i\delta}2R \le M_0^i\Big(1+\frac{1}{\delta}\Big)^{k-i}\Big(\frac{2}{\delta}\Big)^\ell 2R\frac{1}{N^\delta}.\end{aligned} \tag{2.7}$$

Also we parametrize the arcs B with respect to the variable x (say, we use

$$\Big|\int_{-\delta}^0 H(\gamma(x))\gamma'(x)\,dx\Big| \le \int_{-\delta}^0|H(\gamma(x))\gamma'(x)|\,dx$$

and we bound $|\gamma'(x)|\le 3/2$) using formulas (2.2), (2.5) getting

$$\begin{aligned}\Big|\int_B\Big| &\le 2M_0^i\int_{-\delta}^0 N^{-(k-i)x}\Big(\frac{1}{N}+\frac{1}{|x|}\Big)^{k-i}\Big(\frac{2|x|}{R^2}\Big)^\ell N^{kx}\frac{3}{2}\,dx\\
&= 3\frac{2^\ell M_0^i}{R^{2\ell}}\int_0^\delta N^{-ix}\Big(\frac{1}{N}+\frac{1}{x}\Big)^{k-i}x^\ell\,dx \le 3\frac{2^{k+\ell}M_0^i}{R^{2\ell}}\max_{j=0,1,\dots,k-i}\int_0^\delta N^{-ix}\frac{1}{N^{k-i-j}x^j}x^\ell\,dx\\
&\le 3\frac{2^{k+\ell}M_0^i}{R^{2\ell}}\max_{i=1,\dots,k}\int_0^\delta N^{-x}x^{\ell-k+i}\,dx = 3\frac{2^{k+\ell}M_0^i}{R^{2\ell}}O\Big(\frac{1}{\log^2 N}\Big).\end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) in (2.6) gives part (a), recalling that M_0 depends on δ,R,α.

To prove part (b) we set $w=1$ and $R=\log^{2m}N$. Recall that in this case one has $M_0=O(R^{3/2})$ and $\delta=A_1e^{-\frac{\log R}{\log_r R}}$. Then formula (2.8) is (recall $i\le k\le\ell$)

$$O\Big(\frac{M_0^i}{R^{2\ell}\log^2 N}\Big) \le O\Big(\frac{R^{3k/2}}{R^{2\ell}\log^2 N}\Big) \le O\Big(\frac{1}{R^{1/2}\log^2 N}\Big) \le O\Big(\frac{1}{\log^{m+2}N}\Big),$$

and observing that $1/\delta = O(R)$ the term (2.7) is (we will write for short $q = 2m(k3/2+\ell+1)$)

$$O\left(\frac{R^{k3/2+\ell+1}}{N^{\delta}}\right) \leq O\left(\frac{\log^{2m(k3/2+\ell+1)} N}{N^{\delta}}\right) \leq O\left(\frac{\log^q N}{e^{\log N\delta}}\right)$$
$$\leq O\left(\frac{\log^q N}{e^{A_1 e^{\left\{\log\log N - \frac{\log R}{\log_r R}\right\}}}}\right) \leq O\left(\frac{\log^q N}{e^{A_1 e^{\log\log N\{1+o(1)\}}}}\right).$$

Therefore if $i = 1,\dots,k$ then

$$\log^m N J_i \to 0,$$

as $N \to \infty$. Part (b) is proved using the last limit in formula (2.6) observing that

$$\frac{1}{R^{k+\ell-1}} \leq O\Big(\frac{1}{R}\Big) \leq O\Big(\frac{1}{\log^{2m} N}\Big).$$

To prove part (c) we set $w = 1$ and $R = \log^{m_0+\varepsilon} N$. In this case one has $M_0 = O(R^{\lambda})$ and $\delta = A_1 \frac{1}{R^{\beta}}$. Then formula (2.8) is (here $k = i = 1$)

$$O\Big(\frac{M_0}{R^{2\ell}\log^2 N}\Big) \leq O\Big(\frac{1}{R^{2\ell-\lambda}\log^2 N}\Big) \leq O\Big(\frac{1}{\log^{(2\ell-\lambda)(m_0+\varepsilon)+2} N}\Big) \leq O\Big(\frac{1}{\log^{m_0+\varepsilon'} N}\Big),$$

where $\varepsilon' > 0$ and the last identity follows from the hypothesis $m_0(1-\lambda)+2 > 0$ and a suitable ε.

Next observe that $1/\delta = O(R^{\beta})$. Thus the term (2.7) is (here q is some fixed large number)

$$O\left(\frac{\log^q N}{N^{\delta}}\right) \leq O\left(\frac{\log^q N}{e^{\log N\delta}}\right) \leq O\left(\frac{\log^q N}{e^{A_1 \frac{\log N}{\log^{\beta(m_0+\varepsilon)} N}}}\right) \leq O\left(\frac{\log^q N}{e^{A_1 \log^{\varepsilon''} N}}\right)$$

for some positive ε''. This follows from the hypothesis $m_0\beta < 1$ for suitable ε.

Therefore

$$\log^{m_0} N J_1 \to 0$$

as $N \to \infty$. Part (c) is proved using the last limit in formula (2.6) observing that

$$\frac{1}{R^{k+\ell-1}} \leq O\Big(\frac{1}{R}\Big) \leq O\Big(\frac{1}{\log^{m_0+\varepsilon} N}\Big). \qquad \square$$

3. Proof of Corollary 1.5

Assume the hypothesis of Theorem 1.1. We will prove the corollary in two steps.

Claim 1. Assume that the bound

$$F(x+iy) = O(R^{\lambda}), \tag{3.1}$$

holds for $|y| \leq R$ and $1 - A_1 e^{-\frac{\log R}{\log_r R}} \leq x < 1$ (i.e., the growth condition $GC(\lambda)$) and that $|F(x+iy)| = Ae^{e^{c|y|}}$ holds if $1 < x$ (for some positive constants $c < 1, A$). Then the same bound (3.1) holds for the larger region $|y| \leq R$ and $1 - A_1 e^{-\frac{\log R}{\log_r R}} \leq x$.

Proof. In fact Claim 1 follows from a Phragmén–Lindelöf theorem (a rotated version of it):

Let S be the closed half-strip defined by $\Im z \geq 0$ and $1 \leq \Re z \leq 1+\pi$. Assume that $G(z)$ is analytic in an open set containing S and it is bounded on the boundary of S. If there exists positive constants $c < 1, A$ such that $|G(x+iy)| \leq Ae^{e^{cy}}$ whenever $x+iy$ belongs to S then G is bounded on S.

Applying this to $G(z) = \frac{F(z)}{z^{\lambda}}$ yields that $|F(z)| \leq O(R^{\lambda})$ if z belongs to S and $0 \leq y \leq R$. A similar result holds for $\overline{S}$, the set of conjugate points of S. Observe that if $\pi \leq \Re z$ the function $F(z)$ is trivially bounded. The proof of Claim 1 is complete. □

Claim 2. F satisfies the growth condition $GC(3/2)$.

Note that the corollary then follows from Claim 2 and Theorem 1.4 (a).

Proof. The argument which follows is true if R is large enough. Assuming that Claim 1 holds we will show that

$$|F(z)| = O(R^{3\lambda/4}) \tag{3.2}$$

if z belongs to the segment I_R, defined by $z = x+iR$, with $1 - \frac{A_1}{100} e^{-\frac{\log R}{\log_r R}} \leq x \leq 1$ (we have chosen $1/100$ but any small number can be used). In other words, if F satisfies the $GC(\lambda)$ then it satisfies the condition $GC(3\lambda/4)$ (with a new constant A_1', say, instead of A_1 in the definition). Iterating one has that Claim 2 is true.

We recall the following form of the maximum modulus principle due to Lindelöf:

Assume that a function F is analytic on an open set containing a closed disc B with center z_0. Assume that $|F(z)| \leq M$ on the boundary of B and $|F(z)| \leq m$ on an arc of the boundary of B containing an angle of aperture $\frac{2\pi}{3}$. Then $|F(z_0)| \leq M^{2/3} m^{1/3}$.

Hint: One may consider $z_0 = 0$. The maximum of $|F(z)F(ze^{i\pi 2/3})F(ze^{i\pi 4/3})|$ on the boundary of B is bounded by $M^2 m$. Taking $z = 0$ and using the maximum principle the result follows.

To prove (3.2) assume that B is a disc centered at any point of I_R of radius $\frac{A_1}{2} e^{-\frac{\log R}{\log_r R}}$. Using Claim 1 then F is bounded by $A' R^{\lambda} = M$ on the boundary of B, where $A' > 0$ is a fixed constant.

Take the points z on the boundary of B such that $1 + \frac{A_1}{6} e^{-\frac{\log R}{\log_r R}} \leq \Re z$. Such set of points contains an arc of the boundary of B with an angle of aperture at least $\frac{2\pi}{3}$ and on for these points one has $|F(z)| \leq \sum_1^{\infty} \frac{1}{n^{1+\varepsilon}} \leq (1 + \frac{1}{\varepsilon}) = m$, where $\varepsilon = \frac{A_1}{6} e^{-\frac{\log R}{\log_r R}}$. Thus applying the above principle one has

$$F(z) = O(R^{2\lambda/3} e^{\frac{\log R}{3 \log_r R}}) = O(R^{3\lambda/4}).$$

The proof is complete. □

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Departmento e Instituto de Matematica (INMABB), Universidad Nacional del Sur, Av. Alem 1253, 8000 Bahia Blanca, Argentina
E-mail: pablopanzone@hotmail.com