ACTAS DEL XIII CONGRESO
DR. ANTONIO A. R. MONTEIRO (2015)
2016, Páginas 63–79

PRECONTACT RELATIONS AND QUASI-MODAL OPERATORS IN BOOLEAN ALGEBRAS

SERGIO A. CELANI

ABSTRACT. In Boolean algebras the notion of quasi-modal operator is equivalent to the notion of precontact relation. This equivalence offers a new perspective to study the representation of some classes of precontact Boolean algebras. In this article we review some known results on the representation by means of certain filters, called round filters, of the class of compingent Boolean algebras introduced by H. de Vries. The research of these structures using the quasi-modal operators allows the application of algebraic and relational techniques from modal logic. We will show some new characterizations of the maximal round filters (called ends), and we will review the representation theorem proved by H. de Vries for compingent Boolean algebras.

1. INTRODUCTION

This is a survey paper on Boolean algebras endowed with a precontact or subordination relation, but from the viewpoint of quasi-modal operators. We will show that the Boolean algebras endowed with a precontact relation is equivalent to the study of quasi-modal algebras. We will review some known results given by H. de Vries [7] on the representation of compingent Boolean algebras by means of certain filters, called round filters, but using the theory of Boolean algebras endowed with a quasi-modal operator. We will also present some new results on characterization of round filters in some classes of quasi-modal algebras.

In the same way that the Boolean algebra is an abstraction of the powerset of a set, the Boolean algebras endowed with a precontact relation are an abstraction of the proximity spaces [13, 14, 19, 23]. There exist many classes of Boolean algebras endowed with some type of precontact relation. As an example, we can mention the Boolean contact algebras defined in [11], or the Boolean connection algebras defined in [21], or the complete compingent Boolean algebras, also called de Vries algebras, introduced by H. de Vries in [7]. As it is well known, an alternative axiomatization of precontact Boolean algebras can be given via a new relation $\prec$. This relation is known under various names, such as "well inside," "well below," "interior parthood," or "deep inclusion." In [3] the relation $\prec$ is called subordination relation.

The famous Stone Duality Theorem [22] establishes that the category of all zero-dimensional compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category of all Boolean algebras and all Boolean homomorphisms between them. This duality has been generalized in several directions. In 1962, H. de Vries [7] introduced the notion of compingent Boolean algebra and proved that the category of all compact Hausdorff spaces and all continuous maps between them is dually equivalent to the category of de Vries algebras with appropriate morphisms between them. Further refinements of de Vries duality were obtained by Fedorchuk [16] and recently by Bezhanishvili [2] and

2010 *Mathematics Subject Classification.* Primary: 54E05, 03G05; Secondary: 06E99.
This research was partially supported by PIP 112-201101-00636 CONICET.

Dimov [10]. In the topological representation of Boolean algebra the points of the dual Stone space are the ultrafilters of the algebra. However, in the representation given by H. de Vries, the points of the dual space are a particular kind of filters, called maximal round filters or ends. It is easy to see that these filters are a particular class of Δ-filters [4].

It is well known that the variety of modal algebras is the algebraic semantic of classical normal modal logics [20]. The notions of quasi-modal operator and quasi-modal algebra were introduced in [4], as a generalization of the notions of modal operator and modal algebra, respectively. A quasi-modal operator in a Boolean algebra A, is a map Δ that sends each element $a \in A$ to an ideal Δa of A, and satisfies analogous conditions to the modal operator $\Box$ of modal algebras. A quasi-modal algebra is a pair $\langle A, \Delta \rangle$ where A is a Boolean algebra and Δ is a quasi-modal operator. Any modal operator $\Box$ defined in a Boolean algebra A defines a quasi-modal operator $\Delta_{\Box}$ if we put $\Delta_{\Box}(a) = I(\Box a)$, for each $a \in A$. This is the most basic example of quasi-modal operator.

The theory of quasi-modal operators is closely connected with the theory of subordinations, and thus, with the theory of precontact relations. Given a subordination relation $\prec$ in a Boolean algebra A, we can prove that the set $\Delta_{\prec}(b) = \{a \in A : b \prec a\}$ is an ideal of A. So, we can define a map $\Delta_{\prec}$ that sends elements of A to ideals of A. As we shall see, this map is a quasi-modal operator. Conversely, if we have a quasi-modal operator Δ defined in a Boolean algebra A, then the relation $\prec_{\Delta}$ defined by $a \prec_{\Delta} b$ iff $a \in \Delta b$, is a subordination relation on A (for the details see Theorem 15). Thus, we have that the notions of subordination relation and quasi-modal operator are equivalent. This fact has strong consequences, because it puts into evidence that there exists a connection between subordination relations and modal operators in Boolean algebras (see Example 3).

The paper is organized as follows. In Section 2 we shall start recalling some basic definitions and results on quasi-modal operators in Boolean algebras. The majority of the results of this section are in [4], [5] and [6], except for the characterization given in Lemma 9. In Section 3 we will recall the definition of precontact or proximity relation defined in a Boolean algebra [13] [14] [18], and the equivalent notion of subordination relation. In this section we will see that the subordination relations are interdefinable with the quasi-modal operators (Theorem 15). As a consequence, we have that precontac relations, subordinations and quasi-modal operator are equivalent notions. This fact is very important, because these equivalences show a connection between modal logic and precontact structures.

In Section 4 we will study a particular class of filters in quasi-modal algebras, called round filter. The notion of Δ-filters was introduced in [4], and in [5] it was proven that the family of all Δ-filters of a quasi-modal algebra is a lattice dually equivalent to the family of Boolean congruences that preserves, in a certain sense, the quasi-modal operator. The Δ-filters are a generalization of the notion of normal or open filters in modal algebras [20]. Some results of this section are new. For example, in Theorems 28 and 29 we obtain new characterizations of the ends in certain classes of quasi-modal algebras.

In Section 5 we will discuss the representation theory of normal quasi-monadic algebras in terms of maximal round filters. We note that these algebras are equivalent to the compingent Boolean algebras defined in [7]. Most of the results presented in this section are in [7], and are demonstrated by using the notion of subordination relation. Here we use the notion of quasi-modal operator, and we will give some new proofs. We will conclude this section proving de Vries's representation theorem for complete normal quasi-monadic algebras, also called de Vries algebras.

2. QUASI-MODAL ALGEBRAS

We assume that the reader is familiar with basic concepts of distributive lattices and Boolean algebras (see [1] and [17]).

Let $A = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ be a Boolean algebra. By $\mathrm{Ul}(A)$ we shall denote the set of all ultrafilters (or proper maximal filters) of A. By $\mathrm{Id}(A)$ and $\mathrm{Fi}(A)$ we shall denote the families of all ideals and filters of A, respectively. The filter (ideal) generated by a subset $Y \subseteq A$ is denoted by $F(Y)$ $(I(Y))$. If $Y = \{a\}$, then we write $F(a) = [a)(I(a) = (a])$. The set complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$. We recall that $\mathrm{Id}(A)$ is a lattice. We denote the meet and the join in the lattice $\mathrm{Id}(A)$ by $\overline{\wedge}$ and by $\veebar$, respectively,

Let X be a topological space. We denote the closure of Y and the interior of Y in X, by $\mathrm{cl}(Y)$ and by $\mathrm{int}(Y)$, respectively. We recall that a subset U of X is *regular open* if $\mathrm{int}(\mathrm{cl}(U)) = U$. Thus, regular open sets are the interiors of closed sets. The definition of a regular closed set is dual. It is well-known (see, e.g.,[1]) that the collection $\mathrm{RO}(X)$ of regular open subsets of X forms a complete Boolean algebra where $\emptyset$ is the bottom element, X is the top element, and where the operations are defined as follows:

- $-U = \mathrm{int}(X - U)$,
- $U \sqcap V = U \bigcap V$,
- $U \sqcup V = \mathrm{int}(\mathrm{cl}(U \bigcup V))$.

The infinite meets and joins in $\mathrm{RO}(X)$ are given by $\bigwedge U_i = \mathrm{int}(\bigcap U_i)$ and $\bigvee U_i = \mathrm{int}(\mathrm{cl}(\bigcup U_i)$, respectively.

A *modal algebra* is a Boolean algebra A with an operator $\Box : A \to A$ such that $\Box 1 = 1$, and $\Box(a \wedge b) = \Box a \wedge \Box b$, for all $a, b \in A$. Modal algebras provide models of propositional normal modal logics in the same way as Boolean algebras are models of classical logic. In particular, the variety of all modal algebras is the equivalent algebraic semantics of the modal logic **K** in the sense of abstract algebraic logic, and the lattice of its subvarieties is dually isomorphic to the lattice of normal modal logics (for more information on the relation between modal logic and modal algebras see [20])

We recall the notion of quasi-modal operator introduced in [4] (see also [5, 6]).

Definition 1. Let A be a Boolean algebra. A *quasi-modal operator* defined in A is a function $\Delta : A \to \mathrm{Id}(A)$ such that it satisfies the following conditions for all $a, b \in A$:

Q1. $\Delta(a \wedge b) = \Delta a \cap \Delta b$,
Q2. $\Delta 1 = A$.

A pair $\langle A, \Delta\rangle$, where $\Delta : A \to \mathrm{Id}(A)$ is a quasi-modal operator and A is a Boolean algebra, is called a *quasi-modal algebra.* We note that a quasi-modal operator Δ is monotonic, because if $a \leq b$, then $a = a \wedge b$, and so $\Delta a = \Delta(a \wedge b) = \Delta a \cap \Delta b$, i.e., $\Delta a \subseteq \Delta b$.

Example 2. Let A be a Boolean algebra. The map $I_A : A \to \mathrm{Id}(A)$ given by $I_A(a) = (a]$, for each $a \in A$, is clearly a quasi-modal operator on A.

Example 3. Let A be a Boolean algebra. A quasi-modal operator Δ defined in A is called a *principal* if Δa is a principal ideal, for each $a \in A$. In other words, for each $a \in A$, there exists $b \in A$ such that $\Delta a = (b]$. If Δ is principal, then we define a function $\Box_\Delta : A \to A$ as

$$\Box_\Delta(a) = b \text{ iff } \Delta a = (b].$$

Then it is easy to see that $\langle A, \Box_\Delta\rangle$ is a modal algebra.

Conversely, if $\langle A, \Box\rangle$ is a modal algebra, then the map $\Delta_\Box : A \to \mathrm{Id}(A)$ defined by

$$\Delta_\Box(a) = I(\Box a),$$

for each $a \in A$, is a quasi-modal operator. So $\langle A, \Delta_{\Box} \rangle$ is a quasi-modal algebra. Thus, the class of modal algebras can be identified with the class of pairs $\langle A, \Delta \rangle$, where A is a Boolean algebra and Δ is a principal quasi-modal operator.

Example 4. Let X be a nonempty set and let R be a binary relation on X. The pair $\langle X, R \rangle$ is known in modal logic with the name of Kripke frame, or *adjacency spaces* or *precontact space* [8, 23]. Given a Kripke frame $\langle X, R \rangle$, let us take a class $\mathscr{B}$ of subsets of X which form a Boolean algebra under the set-theoretic operations of union $\cup$, intersection $\cap$, and complement $U^c = X - U$, with $U \in \mathscr{B}$, and define a map $\bar{\Delta} : \mathscr{B} \to \mathrm{Id}(\mathscr{B})$ by

$$\bar{\Delta}(U) = \{V \in \mathscr{B} : V \subseteq \Box_R(U)\},$$

where $U \in \mathscr{B}$, and

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}.$$

Then $\bar{\Delta}$ is a quasi-modal operator and thus the pair $\langle \mathscr{B}, \bar{\Delta}_R \rangle$ is a quasi-modal algebra.

Let A be a Boolean algebra. For each Δ quasi-modal operator defined on A we define the dual quasi-modal operator

$$\nabla : A \to \mathrm{Fi}(A)$$

by $\nabla a = \neg \Delta \neg a$, where

$$\neg \Delta x = \{\neg y : y \in \Delta x\}.$$

It is easy to see that the function ∇ satisfies the following conditions:

Q3. $\nabla (a \vee b) = \nabla a \cap \nabla b$,
Q4. $\nabla 0 = A$.

Definition 5. Let $\langle A, \Delta \rangle$ be a quasi-modal algebra. For each $C \subseteq A$ define the following subsets of A:

(1) $\Delta C = I(\bigcup_{c \in C} \Delta c) = \bigvee_{c \in C} \Delta c$,
(2) $\nabla C = F(\bigcup_{c \in C} \nabla c) = \bigvee_{c \in C} \nabla c$,
(3) $\Delta^{-1}(C) = \{a \in A : \Delta a \cap C \neq \emptyset\}$,
(4) $\nabla^{-1}(C) = \{a \in A : \nabla a \subseteq C\}$.
(5) If $C = [a)$, we write $\Delta^{-1}(a)$ instead of $\Delta^{-1}([a))$.

In the following lemma we summarize some important properties.

Lemma 6. *Let $\langle A, \Delta \rangle$ be a quasi-modal algebra.*

(1) $\Delta^{-1}(F) = \bigcup_{a \in F} \Delta^{-1}(a) \in \mathrm{Fi}(A)$, *for each* $F \in \mathrm{Fi}(A)$.
(2) *If* $P \in \mathrm{Ul}(B)$, *then* $\nabla^{-1}(P)^c \in \mathrm{Id}(A)$.
(3) $\Delta I = \bigcup_{a \in I} \Delta a$, *for each* $I \in \mathrm{Id}(A)$.
(4) $\Delta(I_1 \cap I_2) = \Delta(I_1) \cap \Delta(I_2)$, *for all* $I_1, I_2 \in \mathrm{Id}(A)$.
(5) *Let* $P \in \mathrm{Ul}(A)$ *and* $I \in \mathrm{Id}(A)$. *Then*

$$\Delta I \cap P = \emptyset \Leftrightarrow \exists Q \in \mathrm{Ul}(A) \left[\Delta^{-1}(P) \subseteq Q \text{ and } I \cap Q = \emptyset\right].$$

Proof. See [4] and [5]. □

Theorem 7. *Let $\langle A, \Delta \rangle$ be a quasi-modal algebra. Let $a \in A$ and $P \in \mathrm{Ul}(A)$. Then*

(1) $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in \mathrm{Ul}(A) : \Delta^{-1}(P) \subseteq Q$ *then* $a \in Q$,
(2) $a \in \nabla^{-1}(P) \Leftrightarrow \exists Q \in \mathrm{Ul}(A) : Q \subseteq \nabla^{-1}(P)$ *and* $a \in Q$.

Proof. See [4]. □

Let $\langle A,\Delta\rangle$ be a quasi-modal algebra. We define a relation $R_\Delta \subseteq \mathrm{Ul}(A)\times\mathrm{Ul}(A)$ by

$$\begin{aligned}(P,Q)\in R_\Delta &\Leftrightarrow \forall a\in A: \Delta a\cap P\neq\emptyset \text{ then } a\in Q\\ &\Leftrightarrow \Delta^{-1}(P)\subseteq Q.\end{aligned}$$

The relation R_Δ is used in [4] in the representation of quasi-modal algebras. We note that some of these results are similar to the results given in the context of precontact or contact algebras (see [13], [14] and [18]).

Theorem 8. [4] *Let $\langle A,\Delta\rangle$ be a quasi-modal algebra. Then*

(1) *$\Delta a\subseteq I(a)$ is valid in A iff R_Δ is reflexive.*
(2) *$\Delta a\subseteq \Delta^2 a$ is valid in A iff R_Δ is transitive.*
(3) *$I(a)\subseteq\bigcap\{\Delta x : x\in\nabla a\}$ is valid in A iff R_Δ is symmetrical.*
(4) *$\Delta 0=\{0\}$ is valid in A iff R_Δ is serial.*

Let $\langle A,\Delta\rangle$ be a quasi-modal algebra. Following the nomenclature introduced in [4], we shall say that:

(1) $\langle A,\Delta\rangle$ is a *quasi-transitive algebra* if $\Delta a\subseteq\Delta^2 a$, for all $a\in A$,
(2) $\langle A,\Delta\rangle$ is a *quasi-topological algebra* if $\Delta a\subseteq I(a)$ and $\Delta a\subseteq\Delta^2 a$, for all $a\in A$,
(3) $\langle A,\Delta\rangle$ is a *quasi-monadic algebra* if it is a quasi-topological algebra such that $I(a)\subseteq\bigcap\{\Delta x : x\in\nabla a\}$, for all $a\in A$.

So, by Theorem 8, we get that $\langle A,\Delta\rangle$ is a quasi-topological algebra iff the relation R_Δ is reflexive and transitive, and $\langle A,\Delta\rangle$ is a quasi-monadic algebra iff the relation R_Δ is an equivalence.

Now we give another characterization of the property (3) of Theorem 8.

Lemma 9. *Let $\langle A,\Delta\rangle$ be a quasi-modal algebra. Then the following conditions are equivalent:*

(1) *$a\in\Delta b$ implies $\neg b\in\Delta\neg a$;*
(2) *$I(a)\subseteq\bigcap\{\Delta x : x\in\nabla a\}$.*

Proof. (1) $\Rightarrow$ (2). Let $b\leq a$. If $b\notin\bigcap_{x\in\nabla a}\Delta x$, then there exists $x\in\nabla a=\neg\Delta\neg a$ such that $b\notin\Delta x$. We note that $x\in\neg\Delta\neg a$ iff $\neg x\in\Delta\neg a$. Then by assumption $\neg\neg a=a\in\Delta\neg\neg x=\Delta x$, i.e., $a\in\Delta x$, but as Δ is decreasing, $b\in\Delta x$, which is a contradiction.

(2) $\Rightarrow$ (1). Suppose that there are elements $a,b\in A$ such that $a\in\Delta b$ but $\neg b\notin\Delta\neg a$. Then there exists $P\in\mathrm{Ul}(A)$ such that $\Delta\neg a\cap P=\emptyset$ and $\neg b\in P$. By Theorem 7, there exists $Q\in\mathrm{Ul}(A)$ such that $\Delta^{-1}(P)\subseteq Q$ and $\neg a\notin Q$. By Theorem 8, $\Delta^{-1}(Q)\subseteq P$. As $b\notin P$, we have that $\Delta b\cap Q=\emptyset$, but this is a contradiction because $a\in Q$. Thus $\neg b\in\Delta\neg a$. □

Remark 10. (1) Let $\langle A,\Delta\rangle$ be a quasi-modal algebra. It is easy to see that

$$a\in\Delta\Delta b=\Delta^2 b \quad\text{iff}\quad \exists c\in A\ (c\in\Delta b \text{ and } a\in\Delta c). \tag{2.1}$$

(2) If $\langle A,\Delta\rangle$ is a quasi-topological algebra, then the following equation is valid.

$$\Delta a\veebar\Delta b=\Delta(\Delta a\veebar\Delta b). \tag{2.2}$$

Indeed, as $\Delta a\subseteq\Delta a\veebar\Delta b$, and Δ is monotonic we get $\Delta^2 a=\Delta a\subseteq\Delta(\Delta a\veebar\Delta b)$. Similarly, $\Delta b\subseteq\Delta(\Delta a\veebar\Delta b)$. Thus $\Delta a\veebar\Delta b\subseteq\Delta(\Delta a\veebar\Delta b)$. Let $c\in\Delta(\Delta a\veebar\Delta b)=I(\bigcup\{\Delta x : x\in\Delta a\vee\Delta b\})$. Then there exists a finite subset $\{x_1,\ldots,x_n\}\subseteq\Delta a\veebar\Delta b$ and $y_i\in\Delta x_i$, for $1\leq i\leq n$, such that $c\leq y_1\vee\ldots\vee y_n$. As $y_i\in\Delta x_i\subseteq\Delta(x_1\vee\ldots\vee x_n)$, we have that $y_1\vee\ldots\vee y_n\in\Delta(x_1\vee\ldots\vee x_n)$, and consequently $c\in\Delta(x_1\vee\ldots\vee x_n)$. Let $x=x_1\vee\ldots\vee x_n$. So $c\in\Delta x$ and $x\in\Delta a\veebar\Delta b$. As $\Delta x\subseteq I(x)$, we have $c\leq x$, and since $\Delta a\veebar\Delta b$ is an ideal, we deduce that $c\in\Delta a\veebar\Delta b$. Thus (2.2) is valid.

3. PRECONTACT RELATIONS, SUBORDINATION RELATIONS, AND QUASI-MODAL OPERATORS

Standard models of non-discrete theories of space are the contact algebras of regular open subsets of some topological space. In a sense these topological models reflect the continuous nature of the space ([8] [13] [23]). However, in some applications, where digital methods of modeling are used, the continuous models of space are not so suitable. This motivates a search version of the theory of space. One kind of such models are the so called *adjacency spaces*, studied by Düntsch and Vakarelov in [13]. A natural class of Boolean algebras related to adjacency spaces are the precontact algebras, introduced in [13] and [14] under the name of proximity algebras. Now we recall the definition of these structures (see also [9] and [18]).

Definition 11. Let A be a Boolean algebra. A *precontact relation*, or *proximity relation*, defined in A is a relation $\delta \subseteq A \times A$ such that:

P1. If $a\delta b$, then $a \neq 0$ and $b \neq 0$.
P2. $a\delta(b \vee c)$ iff $a\delta b$ or $a\delta c$.
P3. $(a \vee b)\delta c$ iff $a\delta c$ or $b\delta c$.

The pair $\langle A, \delta \rangle$ is called a *precontact Boolean algebra*, or *precontact algebra*, or *Boolean proximity algebra*. These were introduced in [13] as an abstract version of proximity spaces [19]. If $a\delta b$ we say that a is in contact with b, or a is connected to b.

Among others, the following axiomatic extensions of precontact algebras have been studied:

A precontact relation δ is called a *contact relation* if it satisfies the conditions:

P4. If $a\delta b$, then $b\delta a$.
P5. If $a \wedge b \neq 0$, then $a\delta b$.

We will write $a(-\delta)b$ for $(a,b) \notin \delta$. A contact relation δ is called a *Efremovič proximity* if it satisfies the condition:

P6. If $a(-\delta)b$, then there exists $c \in A$ such that $a(-\delta)c$ and $\neg c(-\delta)b$.

If δ is a contact relation on a Boolean algebra A, then the pair $\langle A, \delta \rangle$ is called a Boolean contact algebra, or contact algebra. Examples of precontact and contact relations are based on proximity spaces. We recall the definition of these structures.

Definition 12 ([15])**.** Let X be a set and δ a binary relation on the powerset of X. We call δ a *proximity* on X, and the pair $\langle X, \delta \rangle$ is a *proximity space*, if δ satisfies the following axioms:

(1) $U\delta V$, then $V\delta U$,
(2) $U\delta V$, then $U \neq \emptyset$,
(3) $U \cap V \neq \emptyset$, then $U\delta V$,
(4) $U\delta(V \cup W)$ iff $U\delta V$ or $U\delta W$,
(5) for all $W \in \mathscr{P}(X)$, $U\delta W$ or $V\delta W$, then $U\delta V$.

Clearly, a proximity space $\langle X, \delta \rangle$ produces a Efremovič proximity $\langle \mathscr{P}(X), \delta \rangle$. We say that U is *way below* V, and write $U \prec_\delta V$, whenever $U(-\delta)V^c$. It is not hard to see that the binary relation $\prec_\delta$ defined on $\mathscr{P}(X)$ is interdefinable with δ, because $U\delta V$ iff $U \not\prec_\delta V^c$ (for more details see [19]). So, the theory of proximity spaces can be developed in terms of either δ or $\prec_\delta$. There are many other notions of proximity, and we invite the reader to consult the fundamental text by Naimpally and Warrack [19] or the paper [23] for more examples. The relation $\prec_\delta$ induces the following notion of *subordination* relation [3].

Definition 13. Given a Boolean algebra A a *subordination* defined on A is a binary relation $\prec \subseteq A \times A$ satisfying the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$,
(S2) $a \prec b, c$ implies $a \prec b \wedge c$,
(S3) $a, b \prec c$ implies $a \vee b \prec c$,
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

The relation $\prec$ is also known in the literature under the following names: "well-inside relation," "well below," "interior parthood," "non- tangential proper part," or "deep inclusion." Clearly, the relation $\prec_\delta$ defined by a precontact relation δ on a set X is an example of subordination relation.

There exists a bijective correspondence between precontact relations and subordination relations defined in a Boolean algebra A. If $\delta \subseteq A \times A$ is a precontact relation, then the relation $\prec_\delta$ defined by

$$a \prec_\delta b \text{ iff } a(-\delta)\neg b,$$

is a subordination, where $-\delta$ is the complement of the relation δ.

Conversely, if $\prec \subseteq A \times A$ is a subordination defined in a Boolean algebra A, then the relation $\delta_\prec \subseteq A \times A$ defined by

$$a\delta_\prec b \text{ iff } a \not\prec \neg b,$$

is a precontact relation. The proof of the equivalence of the two definitions is straightforward and analogous to the corresponding statement for proximity spaces (see Theorems 3.9 and 3.11 in [19]). Moreover, $\prec = \prec_{\delta_\prec}$ and $\delta = \delta_{\prec_\delta}$. Therefore we can establish the following result.

Theorem 14. *In a Boolean algebra A the notions of precontact relation and subordination relation are equivalent.*

By this theorem, a precontact algebra $\langle A, \delta \rangle$ can be also defined as a pair $\langle A, \prec \rangle$, where $\prec$ is a subordination relation defined on A. In this case we can say that the pair $\langle A, \prec \rangle$ is a subordination Boolean algebra.

It is not hard to prove that given a subordination relation $\prec$ defined in a Boolean algebra A, the set

$$\prec^{-1}(a) = \{b \in A : b \prec a\}$$

is an ideal of A, for each $a \in A$. In the next theorem we prove that the map $a \to \prec^{-1}(a)$ defines a quasi-modal operator. This easy observation opens the door to study precontact relations or subordinations relations in terms of quasi-modal operators.

Theorem 15. *In a Boolean algebra A the notions of subordination relation and quasi-modal operator are equivalent.*

Proof. Let A be a Boolean algebra. Let $\prec \subseteq A \times A$ be a subordination. For each $a \in A$, take the set

$$\Delta_\prec(a) = \prec^{-1}(a) = \{b \in A : b \prec a\}.$$

It is easy to see that $\Delta_\prec(a)$ is an ideal of A. Thus we have a well defined function $\Delta_\prec : A \to \mathrm{Id}(A)$. Moreover, it is easy to check that $\Delta_\prec$ is a quasi-modal operator.

Reciprocally, let $\Delta : A \to \mathrm{Id}(A)$ be a quasi-modal operator. Define a relation $\prec_\Delta \subseteq A \times A$ by

$$a \prec_\Delta b \text{ iff } b \in \Delta a.$$

Then it is easy to see that $\prec_\Delta$ is a subordination defined on A. Moreover, it is easy to see that $\Delta = \Delta_{\prec_\Delta}$ and $\prec = \prec_{\Delta_\prec}$. □

Corollary 16. *In any Boolean algebra the notions of precontact relation, subordination relation and quasi-modal operator are equivalent.*

As the notions of precontact relations, subordination relations, and quasi-modal operators are equivalent we can take any of them as primitive notion. Instead of precontact relations or subordination relations, we will mainly work with quasi-modal operators. To simplify the notation, we will write $\prec$ instead of $\prec_\Delta$, and δ instead of δ_Δ. Thus, when we refer to a quasi-modal algebra (or precontact algebra) we will write $\langle A,\Delta\rangle$, $\langle A,\delta\rangle$, or $\langle A,\prec\rangle$.

By the previous Corollary 16 we have that certain properties expressed in terms of subordination relations can be expressed in terms of quasi-modal operators. In the following table we will give some known properties, in terms of $\prec$ and Δ:

(S5)	$\Delta a\subseteq I(a)$	$a\prec b$ implies $a\leq b$
(S6)	$I(a)\subseteq\bigcap\{\Delta x : x\in\nabla a\}$	$a\prec b$ implies $\neg b\prec\neg a$
(S7)	$\Delta a\subseteq\Delta^2 a$	$a\prec b$ implies $\exists c\in A\ (a\prec c\prec b)$
(S8)	$\Delta a\neq\{0\}$, when $a\neq 0$	$a\neq 0$ implies $\exists b\neq 0\ (b\prec a)$.

The condition (S7) is called the interpolation axiom, and the condition (S8) is called the extensional axiom.

Now we introduce some classes of quasi-modal algebras that are fundamental in the algebraic study of proximity structures. In what follows, we will be interested in precontact algebras satisfying the additional axiom (S7). As we have said before, these structures were also called quasi-transitive algebras in [4].

Let $\langle A,\Delta\rangle$ be a quasi-modal algebra.

- We shall say that $\langle A,\Delta\rangle$ is a *contact algebra* if Δ satisfies conditions (S5) and (S6) (see [8] and [13]). We recall that these structures are called quasi-topological algebras in [4] and [5].
- We shall say that $\langle A,\Delta\rangle$ is a *normal contact algebra* if $\langle A,\Delta\rangle$ is a contact algebra satisfying the conditions (S5), (S6), (S7), and (S8).

The notion of normal contact algebra was introduced by Fedorchuk [16] under the name of *Boolean δ-algebra* as an equivalent expression of the notion of *compingent Boolean algebra* introduced by de Vries in [7]. A *de Vries* algebra is a *complete* normal algebra $\langle A,\Delta\rangle$, i.e., is a normal quasi-monadic algebra $\langle A,\Delta\rangle$ where A is complete [7].

We give now some examples of precontact and contact algebras. In the following examples we used the notation $\prec$ instead of Δ.

Example 17. Let X be a topological space. The algebra of the regular open subset $\mathrm{RO}(X)$ becomes a contact algebra with respect to the following subordination relation:

$$U\prec_{\mathrm{cl}} V \ \text{ iff } \ \mathrm{cl}(U)\subseteq V,$$

for $U,V\in\mathrm{RO}(X)$. The pair $\langle\mathrm{RO}(X),\prec_{\mathrm{cl}}\rangle$ is a complete contact algebra, called the *standard contact algebra*. We note that $\Delta_{\prec_{\mathrm{cl}}}(U)=\{V\in\mathrm{RO}(X) : V\prec_{\mathrm{cl}} U\}$ is the quasi-modal operator assotiated with $\prec_{\mathrm{cl}}$. Thus the structure $\langle\mathrm{RO}(X),\Delta_{\prec_{\mathrm{cl}}}\rangle$ is a quasi-modal algebra satisfying conditions (S5) and (S6). Also, we can define the precontact relation δ as:

$$U\delta V \ \text{ iff } \ \mathrm{cl}(U)\cap\mathrm{cl}(V)\neq\emptyset.$$

Example 18. Let $\langle X,R\rangle$ be a Kripke frame. As in Example 4, take a class $\mathscr{B}$ of subsets of X which form a Boolean subalgebra of $\mathscr{P}(X)$. Define a relation $\prec_R\subseteq\mathscr{B}\times\mathscr{B}$ as

$$U\prec_R V \ \text{ iff } \ \Box_R(U)\subseteq V,$$

where $U,V\in\mathscr{B}$, and $\Box_R(U)=\{x\in X : R(x)\subseteq U\}$. Then $\prec_R$ is a subordination relation defined on $\mathscr{B}$. Thus $\langle\mathscr{B},\prec_R\rangle$ is a precontact algebra.

Example 19 ([4, 12, 13])**.** Let $\langle X, R\rangle$ be a Kripke frame. Let $\prec_R$ be the subordination relation defined over all subsets of X. Then the conditions below hold:

(1) R is reflexive iff $\prec_R$ satisfies condition (S5).
(2) R is symmetric iff $\prec_R$ satisfies condition (S6).
(3) R is transitive iff $\prec_R$ satisfies condition (S7).

Thus $\langle \mathscr{P}(X), \prec_R\rangle$ is a precontact algebra. Moreover, $\langle \mathscr{P}(X), \prec_R\rangle$ is a contact algebra iff R is reflexive and symmetric. In relation to Example 4, the quasi-modal operator associated with $\prec_R$ is $\bar{\Delta}(U) = \{V \in \mathscr{B} : V \subseteq \Box_R(U)\}$.

4. Round filters and end filters

In this section we study the notion of round filters in a quasi-modal algebra. A round filter is a filter satisfying certain conditions with respect to the quasi-modal operator, and it is a generalization of the notion of normal or open filters in modal algebras [20]. The maximal round filters are very importat in the representation theory developed by H. de Vries in [7], because they are the point of the dual space of a de Vries algebra.

We recall that a filter F in a modal algebra $\langle A, \Box\rangle$ is said to be *open* or *normal* if $\Box a \in F$ when $a \in F$. The importance of open filters is in the fact that they determine the congruences in modal algebras (see [20]). Since a modal algebra can be considered as a quasi-modal algebra where the quasi-modal operator is principal, then we can introduce a generalization of the notion of open filter. In [4] (see also [5]) the following notion was introduced.

Definition 20. Let $\langle A, \Delta\rangle$ be a quasi-modal algebra. A filter F of A is called a Δ-*filter*, if $\Delta a \cap F \neq \emptyset$, provided $a \in F$, i.e., $F \subseteq \Delta^{-1}(F)$. A *round* filter is a Δ-filter F such that $\Delta^{-1}(F) \subseteq F$.

We note that the round filters of $\langle A, \Delta\rangle$ are the fixed points of the function $\Delta^{-1} : \mathrm{Fi}(A) \to \mathrm{Fi}(A)$. In [4] it was proved that the set of all Δ-filters of a quasi-modal algebra $\langle A, \Delta\rangle$ is a lattice. Moreover, the Δ-filters are in bijective correspondence with Boolean congruences that preserve in a certain sense the quasi-modal operator Δ. In the case where the quasi-modal operator is principal (see Example 3), these congruences are Boolean congruences that preserve the modal operator $\Box_\Delta$. For more details see [5].

Let F be a Δ-filter of A. If A satisfies the condition (S5) $\Delta a \subseteq I(a)$, then $\Delta^{-1}(F) = F$. Thus in every quasi-modal algebra satisfying the condition (S5) the notions of round filter and Δ-filter coincide.

We denote the set of all round filters as $\mathrm{Fi}_\Delta(A)$. We call maximal proper round filters *ends*. We denote the set of all ends as $\mathrm{End}(A)$. It is easy to chek (by Zorn's Lemma) that for each $F \in \mathrm{Fi}_\Delta(A)$ there exists $G \in \mathrm{End}(A)$ such that $F \subseteq G$.

Lemma 21. *Let $\langle A, \Delta\rangle$ be a quasi-modal algebra. Let $F \in \mathrm{Fi}(A)$ and $P \in \mathrm{Ul}(A)$. If $\Delta^{-1}(F) \subseteq P$, then there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(Q) \subseteq P$ and $F \subseteq Q$.*

Proof. Let $F \in \mathrm{Fi}(A)$ and $P \in \mathrm{Ul}(A)$. Suppose that $\Delta^{-1}(F) \subseteq P$. Then $F \cap \Delta(P^c) = \emptyset$. So, by the prime filter theorem, there exists $Q \in \mathrm{Ul}(A)$ such that $F \subseteq Q$ and $Q \cap \Delta(P^c) = \emptyset$. So $\Delta^{-1}(Q) \subseteq P$ and $F \subseteq Q$. $\Box$

In the following result we give a characterization of the quasi-modal algebras satisfying the condition (S6).

Theorem 22. *Let $\langle A, \Delta\rangle$ be a quasi-modal algebra. Then the following conditions are equivalent:*

(1) $\langle A, \Delta\rangle$ *satisfies* (S6).

(2) *For all* $F,G \in \mathrm{Fi}(A)$*, if* $\Delta^{-1}(F) \subseteq G$*, then there are* $P,Q \in \mathrm{Ul}(A)$ *such that* $F \subseteq P$*,* $G \subseteq Q$*, and* $\Delta^{-1}(Q) \subseteq P$*.*

Proof. $(1) \Rightarrow (2)$. Let $F,G \in \mathrm{Fi}(A)$. Suppose that $\Delta^{-1}(F) \subseteq G$. As $\neg G = \{\neg a : a \in G\}$ is an ideal, we get that $\Delta(\neg G)$ is also an ideal. Moreover, $F \cap \Delta(\neg G) = \emptyset$. So there exists $P \in \mathrm{Ul}(A)$ such that $F \subseteq P$ and $P \cap \Delta(\neg G) = \emptyset$. We prove that

$$G \cap \Delta(\neg P) = \emptyset.$$

If there exists $g \in G$ such that $g \in \Delta\neg p$, for some $p \in P$, then as A satisfies the condition (S6), we have that $p \in \Delta\neg g$. So $p \in P \cap \Delta(\neg G) = \emptyset$, which is impossible. Thus there exists $Q \in \mathrm{Ul}(A)$ such that $G \subseteq Q$ and $Q \cap \Delta(\neg P) = \emptyset$, i.e., $\Delta^{-1}(Q) \subseteq P$.

$(2) \Rightarrow (1)$. Let $a,b \in A$ such that $a \in \Delta b$ and $\neg b \notin \Delta\neg a$. So $\neg a \notin \Delta^{-1}(\neg b)$. Then there exists $P \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(\neg b) \subseteq P$ and $\neg a \notin P$. So $a \in P$, and there exists $Q \in \mathrm{Ul}(A)$ such that $\neg b \in Q$ and $\Delta^{-1}(P) \subseteq Q$. As $a \in \Delta b \cap P$, $b \in \Delta^{-1}(P) \subseteq Q$. Thus $b, \neg b \in Q$, which is a contradiction. □

Now we give a characterization of the ends in the class of quasi-transitive algebras, but taking into account the notion of quasi-modal operator instead of the notion of subordination. We note that the following two results were proved by H. de Vries in [7] for the class of compingent Boolean algebras.

Lemma 23. *Let* $\langle A,\Delta\rangle$ *be a quasi-transitive algebra. Then*

(1) $\Delta^{-1}(F) \in \mathrm{Fi}_\Delta(A)$*, for all* $F \in \mathrm{Fi}(A)$*.*
(2) *If* $F \in \mathrm{End}(A)$*, then there exists* $U \in \mathrm{Ul}(A)$ *such that* $F = \Delta^{-1}(U)$*.*

Proof. (1) Let $a \in A$ such that $a \in \Delta^{-1}(F)$, i.e., $\Delta a \cap F \neq \emptyset$. We prove that $\Delta a \cap \Delta^{-1}(F) \neq \emptyset$. As $\Delta a \subseteq \Delta^2 a$, by Remark 10 we get that $\Delta^2 a \cap F \neq \emptyset$ iff $\Delta a \cap \Delta^{-1}(F) \neq \emptyset$. So $\Delta^{-1}(F) \in \mathrm{Fi}_\Delta(A)$.

(2) Let $F \in \mathrm{End}(A)$. We consider the family of filters

$$\mathscr{H} = \{H \in \mathrm{Fi}_\Delta(A) : F \subseteq H \quad \text{and } \Delta 0 \cap H = \emptyset\}.$$

As $0 \notin \Delta^{-1}(F) = F$, we have $F \in \mathscr{H}$. So $\mathscr{H} \neq \emptyset$. By Zorn's Lemma, there exists a maximal element U in $\mathscr{H}$. We prove that U is an ultrafilter. Let $a \in A$. Suppose that $a, \neg a \notin U$. Consider the filters $U_a = F(U \cup \{a\})$ and $U_{\neg a} = F(U \cup \{\neg a\})$. So $U_a, U_{\neg a} \notin \mathscr{F}$, i.e., $\Delta 0 \cap U_a \neq \emptyset$ and $\Delta 0 \cap U_{\neg a} \neq \emptyset$. Then there exists $u_1, u_2 \in U$ such that $u_1 \wedge a \in \Delta 0$ and $u_2 \wedge \neg a \in \Delta 0$. Since $u = u_1 \wedge u_2 \in U$, and $\Delta 0$ is an ideal, $(u \wedge a) \vee (u \wedge \neg a) = u \wedge (a \vee \neg a) = u \wedge 1 = u \in \Delta 0$. But this implies that $\Delta 0 \cap U \neq \emptyset$, which is a contradiction. Thus U is an ultrafilter such that $F \subseteq U$, $F = \Delta^{-1}(F) \subseteq \Delta^{-1}(U)$, and $\Delta^{-1}(U)$ is proper. Since $\Delta^{-1}(U) \in \mathrm{Fi}_\Delta(A)$, and F is an end, we get $F = \Delta^{-1}(U)$. □

Theorem 24. *Let* $\langle A,\Delta\rangle$ *be a quasi-transitive algebra satisfying the condition* $\Delta 0 = \{0\}$*. Let* $F \in \mathrm{Fi}(A)$*. Then the following conditions are equivalent:*

(1) $F \in \mathrm{End}(A)$*.*
(2) *For all* $a,b \in A$*, if* $a \in \Delta b$*, then* $\neg a \in F$ *or* $b \in F$*.*

Proof. $(1) \Rightarrow (2)$. Let $a,b \in A$ such that $a \in \Delta b$. Suppose that $\neg a \notin F$. Consider the filter $H = F(F \cup \{a\})$. As $F \subseteq H$, we have that $F = \Delta^{-1}(F) \subseteq \Delta^{-1}(H)$. We note that $\Delta^{-1}(H)$ is proper, because if $0 \in \Delta^{-1}(H)$, then $\Delta 0 \cap H = \{0\} \cap H \neq \emptyset$, i.e., $0 \in H$. Then there exists $f \in F$ such that $f \wedge a = 0$, and consequently $f \leq \neg a$. But this implies that $\neg a \in F$, which is a contradiction. So $\Delta^{-1}(H)$ is a proper filter, and by Lemma 23, $\Delta^{-1}(H) \in \mathrm{Fi}_\Delta(A)$. Since F is maximal, $F = \Delta^{-1}(H)$. We note that $b \in \Delta^{-1}(H)$, because $a \in \Delta b \cap H$. Thus $b \in F$.

$(2) \Rightarrow (1)$. Suppose that there exists a round filter H such that $F \subset H$. So there exists $h \in H - F$. As $H \in \mathrm{Fi}_\Delta(A)$, $\Delta h \cap H \neq \emptyset$, i.e., there exists $b \in \Delta h \cap H$. By assumption, $\neg b \in F$ or $h \in F$. So it must be that $\neg b \in F \subset H$, and as H is closed under $\wedge$ we get that $\neg b \wedge b = 0 \in H$. Therefore, $H = A$, and consequently F is a maximal round filter. □

Definition 25. Let $\langle A, \Delta \rangle$ be a quasi-modal algebra. We shall say that a round filter F is Δ-prime if for all $a, b \in A$, if $(\Delta a \veebar \Delta b) \cap F \neq \emptyset$, then $a \in F$ o $b \in F$.

In the following result we see that in the class of quasi-transitive algebras every end is a Δ-prime round filter.

Lemma 26. *Let $\langle A, \Delta \rangle$ be a quasi-transitive algebra. Then every end is Δ-prime.*

Proof. Let $F \in \mathrm{End}(A)$. Let $a, b \in A$ such that $(\Delta a \veebar \Delta b) \cap F \neq \emptyset$. By Lemma 23, there exists $P \in \mathrm{Ul}(A)$ such that $F = \Delta^{-1}(P)$. Taking into account the identity $\Delta a \veebar \Delta b = \Delta(\Delta a \veebar \Delta b)$, we get the following equivalences:

$$\begin{aligned} (\Delta a \veebar \Delta b) \cap F \neq \emptyset \quad &\text{iff} \quad (\Delta a \veebar \Delta b) \cap \Delta^{-1}(P) \neq \emptyset \\ &\text{iff} \quad \Delta(\Delta a \veebar \Delta b) \cap P \neq \emptyset \\ &\text{iff} \quad (\Delta a \veebar \Delta b) \cap P \neq \emptyset \\ &\text{iff} \quad \Delta a \cap P \neq \emptyset \text{ or } \Delta b \cap P \neq \emptyset \\ &\text{iff} \quad a \in F = \Delta^{-1}(P) \text{ or } b \in F = \Delta^{-1}(P). \end{aligned}$$

□

Now we introduce a class of quasi-modal algebras where the notions of end and Δ-prime round filter coincide.

Definition 27. A quasi-modal algebra $\langle A, \Delta \rangle$ is called a *quasi-pseudo monadic algebra* if it satisfies the following conditions:

(1) $\Delta 0 = \{0\}$,
(2) $\Delta a \veebar \Delta b = \Delta(I(a) \veebar \Delta b)$, for all $a, b \in A$.

We note that if $\langle A, \Delta \rangle$ is a quasi-pseudo monadic algebra, then $\Delta a = \Delta\Delta a$, for all $a \in A$, because

$$\Delta a = \Delta a \veebar \Delta 0 = \Delta(I(0) \veebar \Delta a) = \Delta\Delta a.$$

So A is a quasi-transitive algebra. In the next result we characterize the ends in the class of quasi-pseudo monadic algebras.

Theorem 28. *Let $\langle A, \Delta \rangle$ be a quasi-pseudo monadic algebra. Then a round filter F is an end iff it is Δ-prime.*

Proof. Let F be a round filter. As A is quasi-transitive, by Lemma 26, if F is an end then it is Δ-prime.

Assume that F is Δ-prime. Let $a \in \Delta b$. Then $I(a) \subseteq \Delta b$. So

$$A = I(\neg a) \veebar I(a) \subseteq I(\neg a) \veebar \Delta b,$$

and thus $\Delta(I(\neg a) \veebar \Delta b) = \Delta A = A$. So $\Delta(I(\neg a) \veebar \Delta b) \cap F \neq \emptyset$. As $\Delta(I(\neg a) \veebar \Delta b) = \Delta\neg a \veebar \Delta b$, we have that $\Delta\neg a \veebar \Delta b \cap F \neq \emptyset$. Then $\neg a \in F$ or $b \in F$. □

By Lemma 23 for each round filter F of a quasi-transitive algebra $\langle A, \Delta \rangle$ satisfying the condition $\Delta 0 = \{0\}$ there exists an ultrafilter P of A such that $F = \Delta^{-1}(P)$. On the other hand, a filter of the form $\Delta^{-1}(P)$, with $P \in \mathrm{Ul}(A)$, is a round filter but it is not necessarily an end. Now we prove that $\Delta^{-1}(P)$ is an end when $\langle A, \Delta \rangle$ satisfies the conditions (S5) and (S6).

Theorem 29. *Let* $\langle A,\Delta\rangle$ *be a quasi-transitive algebra satisfying condition* (S5). *Then the following conditions are equivalent:*

(1) $\Delta^{-1}(P)\in \mathrm{End}(A)$, *for each* $P\in \mathrm{Ul}(A)$.
(2) $\langle A,\Delta\rangle$ *satisfies condition* (S6).

Proof. (1) $\Rightarrow$ (2). Let us assume that there exist $a,b\in A$ such that $a\in \Delta b$, but $\neg b\notin \Delta\neg a$. Then there exists $P\in \mathrm{Ul}(A)$ such that $\neg b\in P$ and $\Delta\neg a\cap P=\emptyset$. So $\neg a\notin \Delta^{-1}(P)\in \mathrm{End}(A)$. As $a\in\Delta b$, and $\Delta^{-1}(P)\in \mathrm{End}(A)$, we have that $b\in \Delta^{-1}(P)$. Since $\langle A,\Delta\rangle$ satisfies (S5), we have $\Delta^{-1}(P)\subseteq P$. Then we deduce that $b\in P$, which is impossible, because $\neg b\in P$.

(2) $\Rightarrow$ (1). Let $P\in \mathrm{Ul}(A)$. We apply Theorem 24. Let $a,b\in A$ such that $a\in \Delta b$. As $\Delta b=\Delta^2 b$, by Remark 10 there exists $c\in A$ such that $a\in\Delta c$ and $c\in \Delta b$. If $c\in P$, then $\Delta b\cap P\neq\emptyset$, and consequently $b\in\Delta^{-1}(P)$. If $c\notin P$, then $\neg c\in P$. As $a\in\Delta c$, we deduce that $\neg c\in\Delta\neg a$. So $\Delta\neg a\cap P\neq\emptyset$, i.e., $\neg a\in\Delta^{-1}(P)$. Therefore, $\Delta^{-1}(P)\in\mathrm{End}(A)$. □

For quasi-monadic algebras we have a useful characterization of the ends.

Theorem 30. *Let* $\langle A,\Delta\rangle$ *be a quasi-monadic algebra. Let* $F\in \mathrm{Fi}(A)$. *Then* $F\in\mathrm{End}(A)$ *iff there exists* $P\in\mathrm{Ul}(A)$ *such that* $F=\Delta^{-1}(P)$. *If* $P\in\mathrm{Ul}(A)$, *then it contains a unique end, namely* $\Delta^{-1}(P)$.

Proof. By Lemma 23 and Theorem 29 we get that $F\in\mathrm{End}(A)$ iff there exists $P\in\mathrm{Ul}(A)$ such that $F=\Delta^{-1}(P)$.

We prove that $\Delta^{-1}(P)$ is the unique end such that $\Delta^{-1}(P)\subseteq P$. Suppose that $H\in\mathrm{End}(A)$ and $H\subseteq P$. Then $H=\Delta^{-1}(H)\subseteq\Delta^{-1}(P)$. Suppose that $H\subset\Delta^{-1}(P)$. Then there exists $a\in\Delta^{-1}(P)-H$. So $\Delta a\cap\Delta^{-1}(P)\neq\emptyset$, i.e., there exists $b\in\Delta a\cap\Delta^{-1}(P)$. As H is maximal, by Theorem 24, $\neg b\in H$ or $a\in H$. So it must be $\neg b\in H\subseteq P$. Then $\neg b\in P$. As $b\in\Delta^{-1}(P)\subseteq P$, we get that $b\wedge\neg b=0\in P$, which is impossible. Thus $H=\Delta^{-1}(P)$. □

We finish this section giving a characterization of the condition (S8) in quasi-monadic algebras.

Lemma 31. *Let* $\langle A,\Delta\rangle$ *be a quasi-monadic algebra. Then the following conditions are equivalent:*

(1) *For each* $a\in A-\{0\}$ *there exists* $F\in\mathrm{End}(A)$ *such that* $a\in F$,
(2) $\Delta a\neq\{0\}$, *when* $a\neq 0$.

Proof. (1) $\Rightarrow$ (2). Let $a\neq 0$. Then there exists $F\in\mathrm{End}(A)$ such that $a\in F$. So $\Delta a\cap F\neq\emptyset$, i.e., there exists $b\in A-\{0\}$ such that $b\in\Delta a$. Then $\Delta a\neq\{0\}$.

(2) $\Rightarrow$ (1). If $\Delta a\neq\{0\}$, when $a\neq 0$, then there exists $b\in A-\{0\}$ such that $b\in\Delta a$. As $b\neq 0$ there exists $P\in\mathrm{Ul}(A)$ such that $b\in P$. As $\langle A,\Delta\rangle$ is a quasi-monadic algebra, by Theorem 29 we have that $\Delta^{-1}(P)$ is an end. So $b\in\Delta a\cap P$, i.e., $a\in\Delta^{-1}(P)$. □

5. Representation

In [7, Thm. I.4.5] de Vries showed that given a compact Hausdorff space X, the algebra $\langle \mathrm{RO}(X),\prec_{\mathrm{cl}}\rangle$ is a de Vries algebra (called complete compingent Boolean algebras in [7], or complete normal quasi-monadic algebras according to our nomenclature). Also, de Vries proved that for each de Vries algebra $\langle B,\prec\rangle$ there is a unique, up to homeomorphism, compact Hausdorff space X such that $\langle B,\prec\rangle$ is isomorphic to $\langle\mathrm{RO}(X),\prec_{\mathrm{cl}}\rangle$. As de Vries has shown, this correspondence between compact Hausdorff spaces and de Vries algebras extends to a dual equivalence between the corresponding categories.

Definition 32. We say that a quasi-modal algebra $\langle A, \Delta\rangle$ is *representable* if there is a topological space X and a Boolean embedding $g : A \to \mathrm{RO}(X)$ such that $a \prec_\Delta b$ iff $\mathrm{cl}(g(a)) \subseteq g(b)$, for all $a, b \in A$, where $a \prec_\Delta b$ iff $a \in \Delta b$.

Let $\langle A, \Delta\rangle$ be a quasi-modal algebra. For each $a \in A$, let

$$e(a) = \{F \in \mathrm{End}(A) : a \in F\}.$$

Then we have a Stone-like map $e : A \to \mathscr{P}(\mathrm{End}(A))$.

We denote by **QTB4E** the class of quasi-monadic algebras satisfying the condition (S8). We note that the structures of **QTB4E** are called compingent Boolean algebras in [7].

The next task is to prove that $\{e(a) : a \in A\}$ is a basis for a topology on $\mathrm{End}(A)$ when $\langle A, \Delta\rangle \in$ **QTB4E**.

Proposition 33. *Let $\langle A, \Delta\rangle$ be a quasi-modal algebra. For all $a, b \in A$,*

(1) $e(1) = \mathrm{End}(A)$*;*
(2) $e(a \wedge b) = e(a) \cap e(b)$*;*
(3) $e(a) \cup e(b) \subseteq e(a \vee b)$*;*
(4) *if $\langle A, \Delta\rangle \in$ **QTB4E**, then:*
  (a) $e(a) = \emptyset$ *iff* $a = 0$*;*
  (b) $a \leq b$ *iff* $e(a) \subseteq e(b)$*, thus e is injective.*

Proof. We prove only (4).

(a). If $a = 0$ it is clear that $e(a) = \emptyset$. Assume that $e(a) = \emptyset$. If $a \neq 0$, by Lemma 31, there exists $F \in \mathrm{End}(A)$ such that $a \in F$. So $F \in e(a)$, which is a contradiction. Thus $a = 0$.

(b). As e is meet-preserving, we have that $e(a) \subseteq e(b)$, when $a \leq b$. Assume that $a \nleq b$. Then $a \wedge \neg b \neq 0$. So, by (S8), $\Delta(a \wedge \neg b) \neq \{0\}$. Thus there exists $c \in \Delta(a \wedge \neg b)$ with $c \neq 0$. Consider the filter $\Delta^{-1}(F(c))$. As A is a quasi-monadic algebra, we have by Lemma 23 that $\Delta^{-1}(F(c)) \in \mathrm{Fi}_\Delta(A)$. Moreover $a \wedge \neg b \in \Delta^{-1}(F(c))$. Then there exists $F \in \mathrm{End}(A)$ such that $a \wedge \neg b \in \Delta^{-1}(F(c)) \subseteq F$. As $\neg b \in F$, $b \notin F$. Then $a \in F$ and $b \notin F$. Thus $F \in e(a)$ and $F \notin e(b)$, i.e., $e(a) \nsubseteq e(b)$. □

Lemma 34. *Let $\langle A, \Delta\rangle \in$ **QTB4E**. Then $\{e(a) : a \in A\}$ is a basis for a topology $\mathscr{T}_E$ on* $\mathrm{End}(A)$.

Proof. By Lemma 31, for each $a \neq 0$ there exists $F \in \mathrm{End}(A)$ such that $a \in F$. Thus $\mathrm{End}(A) = \bigcup\{e(a) : a \in A\}$. Moreover, since $e(a \wedge b) = e(a) \cap e(b)$, for all $a, b \in A$, we get that $\{e(a) : a \in A\}$ is a basis for a topology $\mathscr{T}_E$ on $\mathrm{End}(A)$. □

Let $\langle A, \Delta\rangle \in$ **QTB4E**. We shall say that the topological space $\langle \mathrm{End}(A), \mathscr{T}_E\rangle$ is the dual space of $\langle A, \Delta\rangle$.

Proposition 35. *Let $\langle A, \Delta\rangle \in$ **QTB4E** and let $X = \langle \mathrm{End}(A), \mathscr{T}_E\rangle$ be its dual space. Then*

(1) $e(\neg a) = \mathrm{cl}(e(a))^c$*;*
(2) $e(a) = \mathrm{int}(\mathrm{cl}(e(a)))$*, i.e.,* $e(a) \in \mathrm{RO}(X)$*;*
(3) $a \in \Delta B$ *iff* $\mathrm{cl}(e(a)) \subseteq e(b_1) \cup \cdots \cup e(b_n)$*, for some* $\{b_1, \ldots, b_n\} \subseteq B \subseteq A$*; in particular,* $a \in \Delta b$ *iff* $\mathrm{cl}(e(a)) \subseteq e(b)$*;*
(4) $\mathrm{int}(\mathrm{cl}(e(a) \cup e(b))) = e(a \vee b)$*, for all* $a, b \in A$.

Proof. (1). As $\emptyset = e(0) = e(a \wedge \neg a) = e(a) \cap e(\neg a)$, we get that $e(a) \subseteq e(\neg a)^c$. Since $e(\neg a)^c$ is closed, $\mathrm{cl}(e(a)) \subseteq e(\neg a)^c$. We note that as $\{e(a) : a \in A\}$ is a basis, we have that

$$\begin{aligned}\mathrm{cl}(e(a)) &= \bigcap \{e(c)^c : e(a) \subseteq e(c)^c\} \\ &= \bigcap \{e(c)^c : e(a) \cap e(c) = \emptyset\} \\ &= \bigcap \{e(c)^c : e(a \wedge c) = \emptyset\} \\ &= \bigcap \{e(c)^c : a \wedge c = 0\}.\end{aligned}$$

Assume that $F \notin \mathrm{cl}(e(a))$. Then there is $c \in A$ such that $a \wedge c = 0$ and $c \in F$. So, since $c \leq \neg a$, we have that $\neg a \in F$, i.e., $F \in e(\neg a)$.

(2). By (1) we get

$$e(a) = e(\neg\neg a) = \mathrm{cl}(e(\neg a))^c = \mathrm{int}(e(\neg a)^c) = \mathrm{int}(\mathrm{cl}(e(a))).$$

(3). Let $B \subseteq A$. Assume that $a \in \Delta B$. Let $F \in \mathrm{cl}(e(a))$, i.e., $\neg a \notin F$. As $a \in \Delta B$, it is easy to see that there exists $c \in A$ and $b_1, \ldots, b_n \in B$ such that $a \in \Delta c$ and $c \in \Delta b_1 \vee \cdots \vee \Delta b_n$. By Theorem 24, $c \in F$. So $\Delta b_1 \vee \cdots \vee \Delta b_n \cap F \neq \emptyset$. As $F = \Delta^{-1}(P)$, for some $P \in \mathrm{Ul}(A)$, $\Delta b_1 \vee \cdots \vee \Delta b_n \cap \Delta^{-1}(P) \neq \emptyset$, and this implies that

$$\Delta(\Delta b_1 \vee \cdots \vee \Delta b_n) \cap P = \Delta b_1 \vee \cdots \vee \Delta b_n \cap P \neq \emptyset,$$

and thus $\Delta b_i \cap P \neq \emptyset$, for some $1 \leq i \leq n$, i.e, $b_i \in F = \Delta^{-1}(P)$, for some $1 \leq i \leq n$. So $F \in \bigcup_{b \in B} e(b)$.

Conversely, let us assume that $\mathrm{cl}(e(a)) \subseteq \bigcup_{b \in B} e(b)$. By (1), $\bigcap_{b \in B} e(b)^c \subseteq \mathrm{cl}(e(a))^c = e(\neg a)$. Suppose that $a \notin \Delta B$. Then there exists $P \in \mathrm{Ul}(A)$ such that $a \in P$ and $\Delta B \cap P = \emptyset$. So $b \notin \Delta^{-1}(P)$, for every $b \in B$. Since $\Delta^{-1}(P)$ is an end, $\Delta^{-1}(P) \in \bigcap_{b \in B} e(b)^c \subseteq e(\neg a)$. Then $\neg a \in \Delta^{-1}(P)$, and since $\Delta^{-1}(P) \subseteq P$, we get $\neg a \in P$. Therefore, $a \wedge \neg a = 0 \in P$, which is a contradiction. Thus $a \in \Delta B$.

(4). Let $a, b \in A$. Since $e(a) \cup e(b) \subseteq e(a \vee b)$, we have that

$$e(a) \sqcup e(b) = \mathrm{int}(\mathrm{cl}(e(a) \cup e(b))) \subseteq \mathrm{int}(\mathrm{cl}(e(a \vee b))) = e(a \vee b).$$

We prove the inclusion $e(a \vee b) \subseteq \mathrm{int}(\mathrm{cl}(e(a) \cup e(b))$. We note that

$$\mathrm{int}(\mathrm{cl}(e(a) \cup e(b))) = \bigcup \{e(c) : e(c) \subseteq \mathrm{cl}(e(a) \cup e(b))\}.$$

Now, if there exists $c \in A$ such that $e(c) \subseteq \mathrm{cl}(e(a) \cup e(b))$, as

$$\begin{aligned}\mathrm{cl}(e(a) \cup e(b)) &= \mathrm{cl}(e(a)) \cup \mathrm{cl}(e(b)) = e(\neg a)^c \cup e(\neg b)^c \\ &= (e(\neg a) \cap e(\neg b))^c \quad = e(\neg a \wedge \neg b)^c \\ &= e(\neg(a \vee b))^c,\end{aligned}$$

we get that the inclusion $e(c) \subseteq e(\neg(a \vee b))^c$ implies that

$$e(c) \cap e(\neg(a \vee b)) = e(c \wedge \neg(a \vee b)) = \emptyset = e(0).$$

As e is injective, $c \wedge \neg(a \vee b) = 0$, i.e., $c \leq a \vee b$. Thus, if $e(c) \subseteq \mathrm{cl}(e(a) \cup e(b))$, then $c \leq a \vee b$. So

$$\begin{aligned}\mathrm{int}(\mathrm{cl}(e(a) \cup e(b))) &= \bigcup \{e(c) : e(c) \subseteq \mathrm{cl}(e(a) \cup e(b))\} \\ &\subseteq \bigcup \{e(c) : c \leq a \vee b\} = e(a \vee b).\end{aligned}$$

□

Recall that a topological space X is said to be *weakly regular* (see [12]) if X has an open base consisting of regular open sets and for each non-empty open set U there exists a non-empty open set V such that $\mathrm{cl}(V) \subseteq U$.

The following fundamental theorem was first proved by H. de Vries in [7]. Here we give a proof using the theory of quasi-modal operators.

Theorem 36. *Let* $\langle A, \Delta \rangle \in \mathbf{QTB4E}$. *Then* $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ *is a weakly regular,* $T2$*, and compact space.*

Proof. First we will show that $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ is $T2$. If $F, H \in \mathrm{End}(A)$ and $F \neq H$, then there exists a such that $a \in F$ and $a \notin H$. So $\Delta a \cap F \neq \emptyset$, i.e., there exists $b \in \Delta a \cap F$. As H is an end and $a \notin H$, by Theorem 24 we have that $\neg b \in H$. So $F \in e(b)$, $H \in e(\neg b)$, and $e(b) \cap e(\neg b) = e(b \wedge \neg b) = e(0) = \emptyset$. Thus $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ is $T2$.

As each $e(a)$ is a regular open set, we have that $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ is semiregular. Let U be a non-empty open subset of $\mathrm{End}(A)$. So there exists $F \in U$, and as $\{e(a) : a \in A\}$ is a basis for the topology $\mathscr{T}_E$, there exists $a \neq 0$ such that $F \in e(a) \subseteq U$. By the condition (S8), we get that there exists $b \neq 0$ such that $b \in \Delta a$. By Proposition 35, $\mathrm{cl}(e(b)) \subseteq e(a) \subseteq U$. Thus $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ is weakly regular.

We prove that $\mathrm{End}(A)$ is compact. Let $B \subseteq A$ be such that

$$\mathrm{End}(A) = \bigcup \{e(b) : b \in B\}.$$

Consider the ideal $\Delta[B]$. We note that $\Delta[B] = \Delta[I(B)]$. If $\Delta[B]$ is proper, $1 \notin \Delta[B]$. So there exists $P \in \mathrm{Ul}(A)$ such that $\Delta[B] \cap P = \emptyset$. Then $I(B) \cap \Delta^{-1}(P) = \emptyset$, i.e., $b \notin \Delta^{-1}(P)$, for all $b \in B$. As $\Delta^{-1}(P) \in \mathrm{End}(A)$, we have that $\Delta^{-1}(P) \notin \bigcup \{e(b) : b \in B\}$, which is impossible. Thus $\Delta[B]$ is not proper, i.e., $1 \in \Delta[B]$. Then there exists a finite family $\{b_1, \ldots, b_n\}$ of B and there are elements $x_i \in \Delta b_i$, for each $1 \leq i \leq n$, such that $1 = x_1 \vee \cdots \vee x_n$. So $1 = x_1 \vee \cdots \vee x_n \in \Delta b_1 \veebar \cdots \veebar \Delta b_n$. We note that

$$A = \Delta 1 = I(1) = \Delta b_1 \veebar \cdots \veebar \Delta b_n.$$

Let $F \in \mathrm{End}(A)$. Then there exists $P \in \mathrm{Ul}(A)$ such that $F = \Delta^{-1}(P)$. So we have the following equivalences:

$$\begin{aligned}
\Delta 1 \cap \Delta^{-1}(P) \neq \emptyset \quad &\Leftrightarrow \quad (\Delta b_1 \veebar \cdots \veebar \Delta b_n) \cap P \neq \emptyset \\
&\Leftrightarrow \quad \Delta b_i \cap P \neq \emptyset, \text{ for some } 1 \leq i \leq n \\
&\Leftrightarrow \quad \Delta^2 b_i \cap P \neq \emptyset, \text{ for some } 1 \leq i \leq n \\
&\Leftrightarrow \quad \Delta b_i \cap \Delta^{-1}(P) \neq \emptyset, \text{ for some } 1 \leq i \leq n \\
&\Leftrightarrow \quad F = \Delta^{-1}(P) \in e(b_1) \cup \cdots \cup e(b_n).
\end{aligned}$$

So $\mathrm{End}(A) = e(b_1) \cup \cdots \cup e(b_n)$, and consequently the space $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$ is compact. □

Theorem 37. *Let* $\langle A, \Delta \rangle \in \mathbf{QTB4E}$. *Then* $\langle A, \Delta \rangle$ *is representable by means of the topological space* $\langle \mathrm{End}(A), \mathscr{T}_E \rangle$.

Proof. Let $\langle A, \Delta \rangle \in \mathbf{QTB4E}$. Let $X = \langle \mathrm{End}(A), \mathscr{T}_E \rangle$. By Proposition 33 and Theorem 35 we have that the map $e : A \to \mathrm{RO}(X)$ is well defined, is a Boolean embedding, and $a \prec_\Delta b$ iff $\mathrm{cl}(e(a)) \subseteq e(b)$, for all $a, b \in A$. Thus $\langle A, \Delta \rangle$ is representable. □

An *isomorphism* between two de Vries algebras $\langle A, \Delta_A \rangle$ and $\langle B, \Delta_B \rangle$ is a Boolean isomorphism $f : A \to B$ such that

$$a \in \Delta_A b \ \text{ iff } \ f(a) \in \Delta_B f(b),$$

for all $a, b \in A$. We will finish this section proving that if $\langle A, \Delta \rangle$ is a de Vries algebra, then the map e is an isomorphism between $\langle A, \Delta \rangle$ and $\langle \mathrm{RO}(X), \prec_{\mathrm{cl}} \rangle$.

Theorem 38. [7] *If $\langle A, \Delta \rangle$ is a de Vries algebra and X its dual space, then for each $U \in \mathrm{RO}(X)$ there exists $a \in A$ such that $U = e(a)$. So $\langle A, \Delta \rangle$ is isomorphic to $\langle \mathrm{RO}(X), \prec_{\mathrm{cl}} \rangle$.*

Proof. Let $U \in \mathrm{RO}(X)$. As U is open, there exists $B \subseteq A$ such that $U = \bigcup\{e(b) : b \in B\}$. As A is complete, there exists the join $a = \bigvee\{b \in B\}$. It is clear that $e(b) \subseteq e(a)$, for all $b \in B$. Consequently, $U = \bigcup\{e(b) : b \in B\} \subseteq e(a)$.

Now we consider the open set $\mathrm{cl}(U)^c$. So we can write

$$\mathrm{cl}(U)^c = \bigcup\{e(c)^c : c \in C\},$$

for some $C \subseteq A$. So for each $b \in B$ and $c \in C$, we get that $e(b) \subseteq U$ and $e(c) \subseteq \mathrm{cl}(U)^c$. Thus

$$e(b) \cap e(c) = e(b \wedge c) \subseteq U \cap \mathrm{cl}(U)^c = \emptyset = e(0),$$

and since e is injective, $b \wedge c = 0$, i.e., $b \leq \neg c$. Since this holds for all $b \in B$ and $a = \bigvee\{b \in B\}$, we have that $a \leq \neg c$, for all $c \in C$. So $a \wedge c = 0$, for all $c \in C$, and as e is meet-preserving, $e(a \wedge c) = e(a) \cap e(c) = \emptyset$, for all $c \in C$. Then

$$e(a) \subseteq \bigcap\{e(c)^c : c \in C\} = (\bigcup\{e(c)^c : c \in C\})^c = \mathrm{cl}(U).$$

Therefore

$$e(a) = \mathrm{int}(\mathrm{cl}(e(a))) \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{cl}(U))) = \mathrm{int}(\mathrm{cl}(U)) = U,$$

i.e., $e(a) \subseteq U$. Thus $U = e(a)$. Thus e is an isomorphism between $\langle A, \Delta \rangle$ and $\langle \mathrm{RO}(X), \prec_{\mathrm{cl}} \rangle$. □

References

[1] Balbes, R. and Dwinger, P.: *Distributive Lattices*, University of Missouri Press, 1974. MR 0373985.

[2] Bezhanishvili, G.: *Stone duality and Gleason covers through de Vries duality,* Topology Appl. 157 (2010), 1064–1080. MR 2593718.

[3] Bezhanishvili, G., Bezhanishvili, N., Sourabh, S., and Venema, Y.: *Irreducible equivalence relations, Gleason spaces, and de Vries duality,* Appl. Categ. Structures, 2016. DOI: 10.1007/s10485-016-9434-2.

[4] Celani, S. A.: *Quasi-modal algebras,* Math. Bohem. 126 (2001), 721–736. MR 1869464.

[5] Celani, S. A.: *Subdirectly irreducible quasi-modal algebras,* Acta Math. Univ. Comenian. (N.S.) 74 (2005), 219–228. MR 2195481.

[6] Celani, S. A.: *Amalgamation property in quasi-modal algebras,* Rev. Un. Mat. Argentina 50 (2009), 41–46. MR 2643515.

[7] de Vries, H.: *Compact spaces and compactifications. An algebraic approach,* Ph.D. thesis, University of Amsterdam, 1962.

[8] Dimov, G. and Vakarelov, D.: *Contact algebras and region-based theory of space: a proximity approach. I.* Fund. Inform. 74 (2006), 209–249. MR 2284194.

[9] Dimov, G. and Vakarelo, D.: *Topological representation of precontact algebras*, in: Relational methods in computer science, 1–16. Lecture Notes in Comp. Sci., 3929, Springer, Berlin, 2006. MR 2251970.

[10] Dimov, G.: *Some generalizations of Fedorchuk duality theorem—I,* Topology Appl. 156 (2009), 728–746. MR 2492958.

[11] Düntsch, I. and Winter, M.: *Algebraization and representation of mereotopological structures,* J. Relational Methods in Computer Sci. 1 (2004), 161–180.

[12] Düntsch, I. and Winter, M.: *A representation theorem for Boolean contact algebras,* Theoret. Comput. Sci. 347 (2005), 498–512. MR 2187916.

[13] Düntsch, I. and Vakarelov, D.: *Region-based theory of discrete spaces: a proximity approach,* Ann. Math. Artif. Intell. 49 (2007), 5–14. MR 2348380.

[14] Düntsch, I. and Orłowska, E.: *Discrete dualities for some algebras with relations,* J. Log. Algebr. Methods Program. 83 (2014), 169–179. MR 3292915.

[15] Efremovič, V. A.: *The geometry of proximity. I*, Mat. Sbornik (N.S.) 31 (73) (1952), 189–200 (in Russian). MR 0055659.

[16] Fedorčuk, V. V.: *Boolean δ-algebras and quasi-open mappings*, Sibirsk. Mat. Ž. 14 (1973), 1088–1099 (in Russian); English translation: Siberian Math. J. 14 (1973), 759–767 (1974). MR 0341381.

[17] Koppelberg, S.: *Topological duality*, in: Handbook of Boolean algebras, Vol. 1 (J. D. Monk and R. Bonnet, eds), 95–126. North-Holland, Amsterdan, 1989. MR 0991565.

[18] Koppelberg, S., Düntsch, I., and Winter, M.: *Remarks on contact relations on Boolean algebras,* Algebra Universalis 68 (2012), 353–366. MR 3029961.

[19] Naimpally, S. A. and Warrack, B. D.: *Proximity Spaces,* Cambridge University Press, Cambridge, 1970. MR 0278261.

[20] Sambin, G. and Vaccaro, V.: *Topology and duality in modal logic*, Ann. Pure Appl. Logic 37 (1988), 249–296. MR 0934369.

[21] Stell, J.: *Boolean connection algebras: A new approach to the region connection calculus,* Artificial Intelligence 122 (2000), 111–136. MR 1785701.

[22] Stone, M. H.: *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937), 375–481. MR 1501905.

[23] Vakarelov, D.: *Region-based theory of space: Algebras of regions, representation theory, and logics*, in: Mathematical Problems from Applied Logic. II, 267–348, Springer, New York, 2007. MR 2344365.

CONICET AND DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS, UNIVERSIDAD NACIONAL DEL CENTRO, PINTO 399, 7000 TANDIL, ARGENTINA

E-mail: scelani@exa.unicen.edu.ar