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ON THE EXISTENCE OF EXTREMALS FOR THE CRITICAL SOBOLEV IMMERSION WITH VARIABLE EXPONENTS

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ABSTRACT. In this work we review some recent results conserning the existence problem of an extremal for the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ in the critical range, i.e. $\mathscr{A} = \{x \in \Omega\colon q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N-p(x))$ is the critical Sobolev exponent.

1. INTRODUCTION

In this paper we review some recent results on the existence problem for extremals of the Sobolev immersion Theorem for variable exponents $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. By extremals we mean functions where the following infimum is attained

$$S(p(\cdot), q(\cdot), \Omega) := \inf_{v \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla v\|_{p(x),\Omega}}{\|v\|_{q(x),\Omega}}. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded open set and the variable exponent spaces $L^{q(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are defined in the usual way. We refer to the book [5] for the definition and properties of these spaces.

The *critical exponent* is defined as usual by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

When the exponent $q(x)$ is *subcritical*, i.e. $1 \leq q(x) < p^*(x) - \delta$ for some $\delta > 0$, the immersion is compact (see [7, Theorem 2.3]), so the existence of extremals follows easily by direct minimization. But when the subcriticality is violated, i.e. $1 \leq q(x) \leq p^*(x)$ with $\mathscr{A} = \{x \in \Omega\colon q(x) = p^*(x),\ p(x) < N\} \neq \emptyset$ the compactness of the immersion fails and so the existence (or not) of minimizers is not clear. For instance, in the constant exponent case, it is well known that extremals do not exist for any bounded open set Ω.

There are some cases where the subcriticality is violated but still the immersion is compact. In fact, in [18], it is proved that if the criticality set is "small" and we have a control on how the exponent q reaches p^* at the criticality set, then the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact, and so the existence of extremals follows as in the subcritical case.

However, in the general case $\mathscr{A} \neq \emptyset$, up to our knowledge, there are no results regarding the existence or not of extremals for the Sobolev immersion Theorem.

The main importance for the existence of extremals for $S(p(\cdot), q(\cdot), \Omega)$, and of the Sobolev immersion Theorem relays on its connection with the solvability of some nonlinear elliptic PDEs with nonstandard growth, where the so-called $p(x)$-Laplacian is the main example.

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The $p(x)$-Laplacian is defined as $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Observe that for $p(x) \equiv 2$ this operator is the classical Laplace operator Δu and for $p(x) \equiv p$ (constant) is the well known p-Laplace operator that has been widely studied since the 60's.

1.1. **Some history and motivation.** The variable exponent spaces where first considered in W. Orlicz' seminal paper [22] in 1931, but then where left behind as the author pursued the study of the spaces that now bear his name.

The first systematic study of these spaces appeared in H. Nakano's works at the beginning of the 1950s [20, 21] where he developed a general theory in which the spaces $L^{p(x)}(\Omega)$ were a particular example of the more general spaces he was considering. Even though some progress was made after Nakano's work (see in particular the works of the Polish school H. Hudzik, A. Kamińska and J. Musielak in e.g. [12, 13, 19]), it was only in the last 20 years that major progress has been accomplished mainly due to the following facts:

- The discovery of a very weak condition ensuring the boundedness of the Hardy-Littlewood maximal operator in these spaces, i.e. the log-Hölder condition that implies, to begin with, that test functions are dense in $L^{p(x)}(\Omega)$.
- The discovery of the connection of these spaces with the modeling of the so-called *electrorheological fluids* [23].
- The application that variable exponents have shown in image processing [4].

Of central importance in the above mentioned applications are the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ defined as

$$W^{1,p(x)}(\Omega) := \left\{ u \in W^{1,1}_{\text{loc}}(\Omega) \colon u, \partial_i u \in L^{p(x)}(\Omega),\ i = 1, \dots, n \right\},$$

and the subspace of functions with zero boundary values

$$W_0^{1,p(x)}(\Omega) = \overline{\{u \in W^{1,p(x)}(\Omega) \colon u \text{ has compact support}\}},$$

where the closure is taken in the $W^{1,p(x)}(\Omega)$-norm $\|\cdot\|_{1,p(x)}$ that is defined as

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \||\nabla u|\|_{p(x)}.$$

We assume from now on that p is log-Hölder in the sense that

$$\sup_{x,y\in\Omega} |(p(x) - p(y)) \log(|x-y|)| < +\infty. \tag{1.2}$$

Under this assumption it can be proved that the space $C_c^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$ and in $W_0^{1,p(x)}(\Omega)$, and also that the Poincaré inequality holds i.e. there exists a constant $C = C(\Omega, p) > 0$ such that

$$\|u\|_{p(x)} \le C\||\nabla u|\|_{p(x)}$$

for any $u \in W_0^{1,p(x)}(\Omega)$. It follows in particular that $\|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

There are, thus, plenty of literature that deal with the existence problem for nonlinear partial differential equations with the $p(x)$-Laplacian as a main operator (just to cite a few, see [3, 6, 16, 17]).

When the source term has critical growth in the sense of the Sobolev embedding, there are only a handful of results on the existence problem. We refer to the works [8, 11, 18] and also the work [24] where multiplicity results are obtained.

The full proofs of the results presented here can be found in the articles [9, 10].

2. STATEMENT OF THE RESULTS

In order to state our main results, let us introduce some notation.

- The Rayleigh quotient will be denoted by

$$Q_{p,q,\Omega}(v) := \frac{\|\nabla v\|_{p(x),\Omega}}{\|v\|_{q(x),\Omega}}. \tag{2.1}$$

- The Sobolev immersion constant by

$$S(p(\cdot),q(\cdot),\Omega) = \inf_{v\in W_0^{1,p(x)}(\Omega)} Q_{p,q,\Omega}(v). \tag{2.2}$$

- The localized Sobolev constant by

$$S_x = \sup_{\varepsilon>0} S(p(\cdot),q(\cdot),B_\varepsilon(x)\cap\Omega) = \lim_{\varepsilon\downarrow 0} S(p(\cdot),q(\cdot),B_\varepsilon(x)\cap\Omega), \quad x\in\Omega. \tag{2.3}$$

- The critical constant by

$$S = \inf_{x\in\mathscr{A}} S_x. \tag{2.4}$$

- The usual Sobolev constant for constant exponents

$$K(N,p)^{-1} = \inf_{v\in C_c^\infty(\mathbb{R}^N)} \frac{\|\nabla v\|_{p,\mathbb{R}^N}}{\|v\|_{p^*,\mathbb{R}^N}}. \tag{2.5}$$

With these notations, our main results can be stated as

Theorem 2.1. *Assume that $p(\cdot)$, $q(\cdot)\colon\Omega\to[1,+\infty)$ are continuous functions with modulus of continuity $\rho(t)$ such that*

$$\rho(t)\log(1/t)\to 0 \quad as\ t\downarrow 0.$$

Assume, moreover, that the criticality set $\mathscr{A}$ is nonempty. Then, for every domain Ω it holds

$$S(p(\cdot),q(\cdot),\Omega)\le S\le \inf_{x\in\mathscr{A}} K(N,p(x))^{-1}.$$

Theorem 2.2. *Under the same assumptions of the previous Theorem, if* $\sup_\Omega p(\cdot)\le\inf_\Omega q(\cdot)$ *and if the strict inequality holds*

$$S(p(\cdot),q(\cdot),\Omega) < S, \tag{2.6}$$

then there exists an extremal for the immersion $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.

Once this general result is obtained, a natural question is to find conditions on $p(x)$, $q(x)$ and Ω such that (2.6) is satisfied.

First, by employing some rough estimates we obtain *global* conditions that ensure (2.6). More precisely, we get the following result.

Theorem 2.3. *Assume that $B_R\subset\Omega\setminus\mathscr{A}$ where B_R is a ball of radius R. Moreover, assume that* $\sup_{B_R} q<\inf_{B_R} p^*$. *Then, if R is large enough,* (2.6) *is satisfied and therefore there exists an extremal for $S(p(\cdot),q(\cdot),\Omega)$.*

More interesting is to find *local* conditions (in the spirit of the works [1, 2]) that ensure (2.6). In this direction we have

Theorem 2.4. *Let $p(x)$ and $q(x)$ be C^2 exponents such that* $\sup_\Omega p<\inf_\Omega q$. *Assume that there exists $x_0\in\mathscr{A}$ such that $S=S_{x_0}$ and that x_0 is a local minimum of $p(x)$ and a local maximum of $q(x)$, such that either $\Delta p(0)>0$ or $\Delta q(0)<0$. Moreover, assume that $p(x_0)<\sqrt{N}$ if $N\ge 5$, $p(x_0)<2$ if $N=4$ and $p(x_0)<3/2$ if $N=3$. Then* (2.6) *is satisfied and therefore there exists an extremal for $S(p(\cdot),q(\cdot),\Omega)$.*

3. Sketch of the proofs

In this section we briefly indicate the general ideas that lead to the proof of the results.

3.1. **Proof of Theorem 2.1.** The proof of Theorem 2.1 follows from a simple scaling argument. In fact, if we take any $\phi \in C_c^\infty(\Omega)$ and consider the rescaled functions $\phi_\lambda = \lambda^{\frac{-n}{p^*(x_0)}}\phi(\frac{x-x_0}{\lambda})$, it can be checked that our hypothesis on the exponents $p(x)$ and $q(x)$ imply that

$$Q_{p,q,\Omega}(\phi_\lambda) \to \frac{\|\nabla\phi\|_{p(x_0),\mathbb{R}^N}}{\|\phi\|_{p^*(x_0),\mathbb{R}^N}} \qquad \text{as } \lambda \to 0+.$$

So

$$S(p(\cdot),q(\cdot),\Omega) \le \frac{\|\nabla\phi_\varepsilon\|_{p(x),\Omega}}{\|\phi_\varepsilon\|_{q(x),\Omega}} \to \frac{\|\nabla\phi\|_{p(x_0),\mathbb{R}^N}}{\|\phi\|_{p^*(x_0),\mathbb{R}^N}} \qquad \forall\phi \in C_c^\infty(\mathbb{R}^N).$$

Minimizing on ϕ we get

$$S(p(\cdot),q(\cdot),\Omega) \le K(N,p(x_0))^{-1}.$$

Now, taking $\Omega = B_\varepsilon(x_0)$ and minimizing on x_0 and using the monotonicity with repect to the domain, the result of the Theorem follows easily.

3.2. **Proof of Theorem 2.2.** The proof of Theorem 2.2 heavily relies on the Concentration Compactness Principle (CCP) for variable exponents that was proved independently in [8] and [11] and was originally proved for constant exponents in the seminal work of P. L. Lions [14].

The CCP states that given any weak convergent sequence $\{u_k\}_{k\in N} \subset W_0^{1,p(x)}(\Omega)$ with weak limit u, there exist nonnegative measures μ,ν supported in $\bar\Omega$, a countable set I, positive numbers $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I}$ and points $\{x_i\}_{i\in I} \subset \bar\Omega$ such that

$$|u_k|^{q(x)}\,dx \rightharpoonup d\nu = |u|^{q(x)}\,dx + \sum_{i\in I}\nu_i d\delta_{x_i} \tag{3.1}$$

$$|\nabla u_k|^{p(x)}\,dx \rightharpoonup d\mu \ge |\nabla u|^{p(x)}\,dx + \sum_{i\in I}\mu_i d\delta_{x_i} \tag{3.2}$$

$$S_{x_i}\nu_i^{\frac{1}{q(x_i)}} \le \mu_i^{\frac{1}{p(x_i)}}, \tag{3.3}$$

where S_{x_i} is the *localized Sobolev constant* defined in (2.3).

The other key ingredient in the proof is the adaptation of a convexity argument due to P. L. Lions, F. Pacella and M. Tricarico [15].

In fact, what can be proved is that any minimizing sequence either has a strongly convergent subsequence of concentrates around a single point.

To see this, just observe that, by (3.1) if $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ is such that $\|u_k\|_{q(x)} = 1$, then

$$\int_\Omega |u|^{q(x)}\,dx + \sum_{i\in I}\nu_i = 1. \tag{3.4}$$

Now, if in addition $\{u_k\}_{k\in\mathbb{N}}$ is a minimizing sequence for $S(p(\cdot),q(\cdot),\Omega)$, then, by (3.2)

$$1 = \int_\Omega \left|\frac{\nabla u_k}{\|\nabla u_k\|_{p(x)}}\right|^{p(x)} dx \ge \int_\Omega \left|\frac{\nabla u}{S(p(\cdot),q(\cdot),\Omega)}\right|^{p(x)} dx + \sum_{i\in I} S(p(\cdot),q(\cdot),\Omega)^{-p(x_i)}\mu_i + o(1).$$

Now, by Theorem 2.1 and (3.3), we have

$$S(p(\cdot),q(\cdot),\Omega)^{-p(x_i)}\mu_i \ge \nu_i^{\frac{p(x_i)}{q(x_i)}}.$$

On the other hand, assuming that $\|S(p(\cdot),q(\cdot),\Omega)^{-1}\nabla u\|_{p(x)} \le 1$ (the other case is analogous),

$$\int_\Omega \left|\frac{\nabla u}{S(p(\cdot),q(\cdot),\Omega)}\right|^{p(x)} dx \ge \|S(p(\cdot),q(\cdot),\Omega)^{-1}\nabla u\|_{p(x)}^{p^+} \ge \|u\|_{q(x)}^{p^+} \ge \left(\int_\Omega |u|^{q(x)}\,dx\right)^{\frac{p^+}{q^-}},$$

where the Sobolev immersion Theorem has been used.

Combining these last three inequalities, we arrive at

$$\left(\int_\Omega |u|^{q(x)}\,dx\right)^{\frac{p^+}{q^-}} + \sum_{i\in I} \nu_i^{\frac{p(x_i)}{q(x_i)}} \le 1. \tag{3.5}$$

Now, since $p^+ < q^-$ it is easy to see that (3.4) and (3.5) imply that, either $I = \emptyset$ and therefore, $u_k \to u$ strongly in $W_0^{1,p(x)}(\Omega)$, or $u = 0$ and $I = \{i_0\}$, proving the claim.

Finally, it is easy to see that if (2.6) holds, concentration cannot occur. This implies the desired result.

3.3. **Proof of Theorem 2.3.** Theorem 2.3 follows easy by testing $S(p(\cdot),q(\cdot),\Omega)$ with any function $u_R(x) = u(x/R)$ with $u \in C_c^\infty(B_1)$ (assuming that $B_R \subset \Omega \setminus \mathscr{A}$) and observing that $Q_{p(x),q(x),\Omega}(u_R) \to 0$ as $R \to \infty$.

3.4. **Proof of Theorem 2.4.** This theorem is more subtle. The idea is first to show that under the considered hypotheses one has that $\bar{S} = K(N,p(x_0))^{-1}$ and then evaluate $Q_{p(x),q(x),\Omega}$ in a properly rescaled extremal for $K(N,p(x_0))^{-1}$. Then, a fine asymptotic analysis shows that if the test function u_ε is concentrated enough one has

$$Q_{p(x),q(x),\Omega}(u_\varepsilon) < K(N,p(x_0))^{-1} = \bar{S}$$

and so (2.6) holds.

References

[1] Thierry Aubin. *Problèmes isopérimétriques et espaces de Sobolev*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 5, A279–A281. MR 0407905.

[2] Haïm Brézis and Louis Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477. MR 0709644.

[3] Alberto Cabada and Rodrigo L. Pouso, *Existence theory for functional p-Laplacian equations with variable exponents*, Nonlinear Anal. **52** (2003), no. 2, 557–572. MR 1937640.

[4] Yunmei Chen, Stacey Levine, and Murali Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406 (electronic). MR 2246061.

[5] Lars Diening, Petteri Harjulehto, Peter Hästö, and Michael Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011. MR 2790542.

[6] Xian-Ling Fan and Qi-Hu Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), no. 8, 1843–1852. MR 1954585.

[7] Xianling Fan and Dun Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446. MR 1866056.

[8] Julián Fernández Bonder and Analía Silva, *Concentration-compactness principle for variable exponent spaces and applications*, Electron. J. Differential Equations (2010), no. 141, 18. MR 2729462.

[9] Julián Fernández Bonder, Nicolas Saintier, and Analia Silva. *On the Sobolev embedding theorem for variable exponent spaces in the critical range.* J. Differential Equations **253** (2012), no. 5, 1604–1620. MR 2927392.

[10] Julián Fernández Bonder, Nicolas Saintier, and Analia Silva. *Existence of solution to a critical equation with variable exponent.* Ann. Acad. Sci. Fenn. Math. **37** (2012), 579–594. MR 2987088.

[11] Yongqiang Fu, *The principle of concentration compactness in $L^{p(x)}$ spaces and its application*, Nonlinear Anal. **71** (2009), no. 5-6, 1876–1892. MR 2524401.

[12] Henryk Hudzik. On generalized Orlicz-Sobolev space. *Funct. Approximatio Comment. Math.* **4** (1976), 37–51. MR 0442671.

[13] Anna Kamińska. Flat Orlicz-Musielak sequence spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **30** (1982), no. 7-8, 347–352. MR 0707748.
[14] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201. MR 0834360.
[15] P.-L. Lions, F. Pacella, and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), no. 2, 301–324. MR 0963504.
[16] Mihai Mihăilescu, *Elliptic problems in variable exponent spaces*, Bull. Austral. Math. Soc. **74** (2006), no. 2, 197–206. MR 2260488.
[17] Mihai Mihăilescu and Vicenţiu Rădulescu, *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proc. Amer. Math. Soc. **135** (2007), no. 9, 2929–2937. MR 2317971.
[18] Yoshihiro Mizuta, Takao Ohno, Tetsu Shimomura, and Naoki Shioji, *Compact embeddings for Sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)$-Laplacian and its critical exponent*, Ann. Acad. Sci. Fenn. Math. **35** (2010), no. 1, 115–130. MR 2643400.
[19] Julian Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983. MR 0724434.
[20] Hidegorô Nakano. *Modulared semi-ordered linear spaces*. Maruzen Co. Ltd., Tokyo, 1950. MR 0038565.
[21] Hidegorô Nakano. *Topology of linear topological spaces*. Maruzen Co. Ltd., Tokyo, 1951. MR 0046560.
[22] Władysław Orlicz. Uber konjugierte Exponentenfolgen. *Studia Mathematica* **3** (1931), no. 1, 200–211.
[23] Michael Ružička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000. MR 1810360.
[24] Analia Silva, *Multiple solutions for the $p(x)$-Laplace operator with critical growth*, Adv. Nonlinear Stud. **11** (2011), no. 1, 63–75. MR 2724542.

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