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ON THE CONVERGENCE OF SOME CLASSES OF DIRICHLET SERIES

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ABSTRACT. We present a brief and informal account on the so-called *Bohr's absolute convergence problem* on Dirichlet series, from its statement and solution in the beginnings of the 20th century to some of its recent variations.

INTRODUCTION

In London, during the 1908 Olympic Games, Denmark defeated France 17-1 in one of the football (a.k.a. soccer) semifinals. This remains an Olympic record to this day. Denmark eventually lost the final 2-0 against Great Britain. Two years later, the whole Danish national team attended the PhD dissertation "Bidrag til de Dirichletske Rækkers Theori" (Contributions to the Theory of Dirichlet Series), at the University of Copenhagen. Dirichlet series is, of course, a subject that has always attracted the attention of football players, but the main reason for them to be there was that the one presenting the thesis was Harald Bohr, a fine midfielder of Akademisk Boldklub and one of the stars of the national team. He was one of the most popular sporstmen in Denmark at the moment. In the audience was also the goalkeeper of Akademisk Boldklub, Niels, the older brother of Harald. In 1922 he won the Nobel Prize in Physics, but he never got to play in the national team...

In his thesis Bohr dealt with Dirichlet series, studying them from the point of view of holomorphic functions on one variable and linking them with power series in infinitely many variables. He was mainly interested in convergence of Dirichlet series and stated what later was to be known as *Bohr's absolute convergence problem*. In this note we give a short and informal account of Bohr's absolute convergence problem for Dirichlet series: statement, solution and some late variations of the problem.

1. CONVERGENCE OF DIRICHLET SERIES

A Dirichlet series is a formal series $D = D(s)$ of the form

$$D = \sum_n a_n \frac{1}{n^s}$$

with coefficients $a_n \in \mathbb{C}$ and variable s in some region of $\mathbb{C}$. Of course, the most famous Dirichlet series is $\zeta(s) = \sum_n \frac{1}{n^s}$. Dirichlet series and power series are very much related through the theory of general Dirichlet series, of which both are particular cases.

The convergence of power series is a very well understood issue and is part of the background knowledge of every mathematician. If a power series converges (or converges absolutely) at some $z_0 \in \mathbb{C}$, then it converges (or converges absolutely) for every $z \in \mathbb{C}$ with $|z| < |z_0|$. Then the natural domains to think of convergence of power series are disks and it

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makes sense to think of radius of convergence and absolute convergence. As we know, these two radii coincide, and it turns out to be the supremum of all radii of uniform convergence. Moreover, in the open disk of convergence the power series defines a holomorphic funcion which is bounded in any smaller disk (since it is the uniform limit of polynomials).

The situation for Dirichlet series turns out to be quite different. If a Dirichlet series D converges (or converges absolutely) at some $s_0 \in \mathbb{C}$, then it converges (or converges absolutely) at every $s \in \mathbb{C}$ with $\operatorname{Re} s > \operatorname{Re} s_0$. This means that, while disks are the regions of convergence of power series, half-planes of the form $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$ are the regions of convergence of Dirichlet series, and instead of radii we now have the abscissas of convergence and of uniform convergence:

$$\sigma_c(D) = \inf\{\sigma : \text{the series } D \text{ converges in } \operatorname{Re} s > \sigma\},$$
$$\sigma_a(D) = \inf\{\sigma : \text{the series } D \text{ converges absolutely in } \operatorname{Re} s > \sigma\}.$$

A natural question now is if these two abscissas are equal. This can be easily answered in the negative, simply by considering $D = \sum_n (-1)^n \frac{1}{n^s}$; we obviously have $\sigma_a(D) = 1$ and, by Leibniz's criterion for alternate series, $\sigma_c(D) = 0$. Moreover, we can show that this is the farthest apart that these two abscissas can be: indeed, if D converges at s_0 then we have

$$\sum_n \frac{|a_n|}{|n^{s_0+1+\varepsilon}|} = \sum_n \frac{|a_n|}{n^{\operatorname{Re} s_0+\varepsilon/2}} \frac{1}{|n^{1+\varepsilon/2}|} < \infty.$$

Hence, $\sigma_a(D) \leq \sigma_c(D) + 1$ for every Dirichlet series D. This means that the maximum width of the strip where a Dirichlet series converges not absolutely is 1 or, more precisely,

$$\sup\{\sigma_a(D) - \sigma_c(D) \colon D \text{ Dirichlet series}\} = 1.$$

On the set $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_c(D)\}$ the series defines a holomorphic function. The main interest of Bohr was to be able to determine $\sigma_a(D)$ from the analytic properties of this function. He then considered the abscissa of boundedness

$$\sigma_b(D) = \inf\{\sigma \colon \text{the series defines a bounded holomorphic function in } \operatorname{Re} s > \sigma\}$$

and also the abscissa of uniform convergence

$$\sigma_u = \inf\{\sigma : \text{ the series converges uniformly in } \operatorname{Re} s > \sigma\}.$$

We easily have

$$\sigma_c(D) \leq \sigma_b(D) \leq \sigma_u(D) \leq \sigma_a(D).$$

Then Bohr's aim was to try to distinguish between these abscissas. First of all, by a fundamental theorem of Bohr [5, Satz 1] we have that for every Dirichlet series D

$$\sigma_b(D) = \sigma_u(D). \tag{1}$$

The so-called *Bohr's absolute convergence problem* asks for the maximum width of the strip where a Dirichlet series converges uniformly but not absolutely. More precisely, let us define

$$S := \sup\left\{\sigma_a(D) - \sigma_u(D) \,:\, D = \sum_n a_n \frac{1}{n^s} \text{ Dirichlet series}\right\}.$$

The problem is then to determine the exact value of S.

Let us reformulate S in a way that is more keen to functional analysis. First, Bohr's fundamental theorem (1), a translation argument and standard manipulations of suprema give

$$\begin{aligned} S &= \sup\left\{\sigma_a(D)-\sigma_u(D) \,:\, D=\sum_n a_n\frac{1}{n^s} \text{ Dirichlet series}\right\} \\ &= \sup\left\{\sigma_a(D) \,:\, \text{Dirichlet series with } \sigma_b(D)=\sigma_u(D)=0\right\} \\ &= \sup\left\{\sigma_a(D) \,:\, \text{Dirichlet series bounded on } \mathrm{Re}\, s>0\right\}. \end{aligned} \tag{2}$$

We now define $\mathscr{H}_\infty$ as the vector space of all Dirichlet series $D=\sum_n a_n n^{-s}$ such that $\sigma_c(D)\le 0$ and that the limit function $D(s)=\sum_n a_n \frac{1}{n^s}$ is bounded on $\mathrm{Re}\, s>0$.

It can be seen that $\mathscr{H}_\infty$ is a Banach space with the supremum norm given by

$$\Big\|\sum_n a_n n^{-s}\Big\|_{\mathscr{H}_\infty} = \sup_{\mathrm{Re}\, s>0}\Big|\sum_{n=1}^{\infty} a_n\frac{1}{n^s}\Big|.$$

Now (2) can be rewritten as

$$S=\sup\{\sigma_a(D) \,:\, D\in\mathscr{H}_\infty\}. \tag{3}$$

On the other hand from (1) and a simple translation argument we immediately have

$$\sigma_u(D)=\inf\Big\{\sigma\in\mathbb{R} \;\Big|\; \sum_n \frac{a_n}{n^\sigma}\frac{1}{n^s}\in\mathscr{H}_\infty\Big\}. \tag{4}$$

Another brilliant idea of Bohr was to look at the problem in a totally different way: instead of working with Dirichlet series in one complex variable, work with power series in infinitely many variables. He identified each Dirichlet series with a power series in infinitely many variables as follows.

Let $p=(p_1,p_2,p_3,\dots)$ be the sequence of prime numbers. For each multi-index $\alpha=(\alpha_1,\dots,\alpha_N,0,\dots)\in\mathbb{N}_0^{(\mathbb{N})}$ we set

$$p^\alpha = p_1^{\alpha_1}\times\cdots\times p_N^{\alpha_N}.$$

We have a one-to-one correspondence

$$\alpha\in\mathbb{N}_0^{(\mathbb{N})} \leftrightsquigarrow n\in\mathbb{N} \qquad \text{where } p^\alpha=n. \tag{5}$$

Given a sequence of complex numbers $z=(z_1,z_2,z_3,\dots)$ and a multi-index $\alpha\in\mathbb{N}_0^{(\mathbb{N})}$ we write

$$z^\alpha = z_1^{\alpha_1}\times\cdots\times z_N^{\alpha_N}. \tag{6}$$

Then a formal power series in infinitely many variables is an expression of the form

$$\sum_\alpha c_\alpha z^\alpha,$$

where the sum is over all multi-indexes $\alpha\in\mathbb{N}_0^{(\mathbb{N})}$ and each c_α is a complex number. We denote by $\mathfrak{P}$ the set of all formal power series and $\mathfrak{D}$ the set of all formal Dirichlet series (note that we are not assuming any kind of convergence whatsoever). Then the relation (5) defines a mapping $\mathfrak{B}$, which will be called the *Bohr transform*:

$$\begin{array}{cccc} \mathfrak{B}: & \mathfrak{P} & \longrightarrow & \mathfrak{D} \\ & \sum_\alpha c_\alpha z^\alpha & \xrightarrow{c_\alpha=a_{p^\alpha}} & \sum_n a_n\frac{1}{n^s} \end{array}$$

This is easily checked to be a bijective algebra homomorphism. Then, every problem on the side of Dirichlet series has an immediate translation in the side of power series. There should then be a problem stated in terms of absolute convergence of power series in infinitely many variables that corresponds to Bohr's absolute convergence problem. Let us find out what it is.

Our aim was to get closer to functional analysis and we rephrased the problem in (3) involving a Banach space of Dirichlet series. Our first goal now is to describe the pre-image of this Banach space through the Bohr transform. Let us begin by noting that power series on finitely many variables define holomorphic functions, and the natural domains of convergence of such series are polydisks. Then the natural norm to consider on $\mathbb{C}^N$ in this context is the sup–norm (the unit ball being the unit polydisk), rather than the Euclidean norm. Then, when jumping to infinitely many variables, it is reasonable to consider the space c_0 (which is endowed with the sup–norm) and take its open unit ball

$$B_{c_0} = \big\{ z = (z_1, z_2, \dots) \colon |z_j| < 1 \text{ for all } j \text{ and } |z_j| \to 0 \big\}$$

as the natural setting to consider power series in infinitely many variables.

A function $f : B_{c_0} \to \mathbb{C}$ is holomorphic if it is Fréchet differentiable at every point. Each holomorphic function on B_{c_0} has an associated family of coefficients: given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N, 0, 0, \dots)$ consider f_N the restriction to $\mathbb{C}^N \times \{0\}$, which is a holomorphic function in N variables and hence has a monomial series expansion; take then $c_\alpha(f) = c_\alpha(f_N)$. Then holomorphic functions on B_{c_0} define formal power series (we write $f \sim \sum_\alpha c_\alpha(f) z^\alpha$). Then the Banach space

$$H_\infty(B_{c_0}) = \{ f : B_{c_0} \to \mathbb{C} \colon f \text{ is holomorphic and bounded} \}$$

(endowed with the norm $\|f\| = \sup_{z \in B_{c_0}} |f(z)|$) can be seen as a subset of $\mathfrak{P}$. Then the ideas of Bohr give the following fundamental theorem.

Theorem 1 (see Bohr [5], and also Hedenmalm-Lindqvist-Seip [15]). *Bohr's transform $\mathfrak{B}$ is an isometric isomorphism between $H_\infty(B_{c_0})$ and $\mathscr{H}_\infty$.*

Why could such a result be of any use to Bohr in order to face his absolute convergence problem? To understand this let us point out that, unlike power series in finitely many variables that converge at every point on a certain polydisk, power series of holomorphic functions on B_{c_0} do not necessarily converge. This was shown by Toeplitz, who gave in [17] an example of a function in $H_\infty(B_{c_0})$ with a power series such that for every $\varepsilon > 0$ there exists $z \in \ell_{4+\varepsilon} \cap B_{c_0}$ for which the power series does not converge absolutely. So then a natural question now is to ask for which z's do we have that the formal power series of every function in $H_\infty(B_{c_0})$ converge.

Take $f \in H_\infty(B_{c_0})$ and let $\sum_\alpha c_\alpha(f) z^\alpha$ be its associated power series. If $z \in B_{c_0}$ satisfies

$$\sum_{j=1}^{\infty} |z_j| < \infty,$$

then we can show that

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)| |z|^\alpha < \infty.$$

In other words, for $z \in \ell_1 \cap B_{c_0}$, the power series of every function in $H_\infty(B_{c_0})$ is absolutely convergent at z. What happens if we change ℓ_1 for some other ℓ_r? Let us define

$$M = \sup \Big\{ r : \sum_\alpha |c_\alpha(f)| |z|^\alpha < \infty \text{ for } z \in \ell_r \cap B_{c_0} \text{ and } f \in H_\infty(B_{c_0}) \Big\}.$$

Then Toeplitz's example and what we have just noted give $1 \le M \le 4$.

Then the power series counterpart of Bohr's absolute convergence problem is given by the following deep result of Bohr [4, Satz IX]:

$$S = \frac{1}{M}.$$

Working with power series, Bohr was able to show [4, Satz III] that $M \geq 2$. On the other hand, as we have already mentioned, Toeplitz's example gives $M \leq 4$. This gives $1/4 \leq S \leq 1/2$, and that is what was known on the subject in 1913... and so it remained for some time.

Let us take a systematic approach to the problem. Let $1 \leq q < \infty$, then Hölder's inequality gives:

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha||z|^\alpha \leq \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha|^q \right)^{1/q} \left(\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} (|z|^{q'})^\alpha \right)^{1/q'}$$

It can be seen that the last series converges if and only if $z \in \ell_{q'}$ (note that the α's are multi-indexes). This then gives the following.

Theorem 2. *Let $\mathscr{F} \subseteq H_\infty(B_{c_0})$ be such that the coefficients of all $f \in \mathscr{F}$ are q-summable. Then, $\sum_\alpha |c_\alpha(f)|\,|z|^\alpha < \infty$ for every $z \in \ell_{q'} \cap B_{c_0}$ and every $f \in \mathscr{F}$.*

One could naïvely expect that for some q a result like the following one holds:

The coefficients of any function in $H_\infty(B_{c_0})$ are q-summable.

This would imply $M \geq q'$, and then $S \leq 1/q' = 1 - 1/q$. Furthermore if we were able to find the optimal value of q, this would give us the equality. Unfortunately, such a result does not hold...

However, a similar result holds if we just consider homogeneous polynomials. That was the (not naïve at all) approach of Bohnenblust and Hille [3], who finally settled the problem in 1931. In order to present their results, some definitions are needed.

Given an m-linear map $A : c_0 \times \cdots \times c_0 \to X$, the mapping

$$P\colon c_0 \to X \quad \text{given by} \quad P(z) = A(z, \ldots, z)$$

is called an m-homogeneous polynomial. Of all m-linear maps defining a polynomial P, there exists only one that is symmetric, which we denote by $\overset{\vee}{P}$.

If we write each $z \in c_0$ as $z = \sum_i z_i e_i$, then we have

$$P(z) = \overset{\vee}{P} \left(\textstyle\sum_i z_i e_i, \ldots, \sum_i z_i e_i \right) = \sum_{i_1} \cdots \sum_{i_m} \overset{\vee}{P} (e_{i_1}, \ldots, e_{i_m}) z_{i_1} \cdots z_{i_m}.$$

So, every m-homogeneous polynomial defines a family of coefficients through the associated m-linear mapping. On the other hand, an m-homogeneous polynomial is a holomorphic mapping defined on c_0 and hence defines a formal power series

$$P \sim \sum_{\alpha_1 + \alpha_2 + \cdots = m} c_\alpha(P) z^\alpha .$$

Both families of coefficients $\left(\overset{\vee}{P} (e_{i_1}, \ldots, e_{i_m}) \right)_{i_1, \ldots, i_m}$ and $\left(c_\alpha(P) \right)_\alpha$ are closely related, but different. In fact, for $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots)$ one has

$$c_\alpha(P) = \frac{m!}{\alpha_1! \cdots \alpha_n!} \overset{\vee}{P} (e_1, \overset{\alpha_1}{\ldots}, e_1, e_2, \overset{\alpha_2}{\ldots}, e_2, \ldots, e_n, \overset{\alpha_n}{\ldots}, e_n) . \tag{7}$$

Bohnenblust and Hille proved in [3] a fundamental result on the coefficients of a polynomial.

Theorem 3 (Bohnenblust–Hille inequality)**.** *There is a constant $C > 0$ such that for each m-homogeneous polynomial $P : c_0 \to \mathbb{C}$ we have*

$$\Big(\sum_{i_1,\dots,i_m} | \overset{\vee}{P} (e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq C^m \sup_{z \in B_{c_0}} |P(z)|$$

and the exponent is optimal.

As an immediate consequence of this inequality and the optimality of the exponent, Bohr's absolute convergence problem is finally solved.

Corollary 4 (Bohr–Bohnenblust–Hille)**.**

$$S = 1/2 = \lim_{m \to +\infty} 1 - \frac{m+1}{2m}.$$

Let us note that from Theorem 3 and (7) we easily have that the coefficients $\big(c_\alpha(P)\big)_\alpha$ are $\frac{m+1}{2m}$-summable. Then since for $q = \frac{2m}{m+1}$ we have $1/q' = 1 - \frac{m+1}{2m}$, the spirit of Theorem 2, which could be summarized as: *if coefficients are q-summable, then S has to do with* $1/q'$ lays behind this result.

2. Vector-valued setting

An expression of the form

$$D = \sum_n a_n \frac{1}{n^s}$$

makes sense when the coefficients a_n belong to a Banach space X. Again, convergence takes place in half-planes and it makes sense to ask about the maximal width of the band on which a Dirichlet series may converge uniformly but not absolutely. Using Hahn-Banach Theorem and working a little bit one can see that Bohr's fundamental theorem, i.e. $\sigma_u = \sigma_b$, holds also in the vector valued case [9, Proposition 2]. Therefore it is natural to define $\mathscr{H}_\infty(X)$ as the space of all Dirichlet series D with coefficients in X which define bounded and holomorphic functions on $\mathrm{Re}\, s > 0$. The question is, now, to determine

$$S(X) = \sup \{ \sigma_a(D) \, : \, D \in \mathscr{H}_\infty(X) \} . \tag{8}$$

The Bohnenblust–Hille inequality (Theorem 3) is one of the main ingredients in the final solution of Bohr's problem in the scalar case (Corollary 4). We will need then an analogous result for vector-valued polynomials. But before we go into that question let us recall that a Banach space has cotype q (see e.g. [12, Chapter 11]) if there exists a constant $C > 0$ such that for every finite choice of elements $x_1, \dots, x_N \in X$

$$\Big(\sum_{k=1}^N \|x_k\|^q \Big)^{1/q} \leq C \bigg(\int_0^1 \Big\| \sum_{k=1}^N r_k(t) x_k \Big\|^2 dt \bigg)^{1/2} ,$$

where r_k is the k-th Rademacher function. We will denote $q(X)$ for the infimum over all q's such that X has cotype q. This number is well known for many classical Banach spaces, for example, if $X = L_p(\mu)$, then

$$q(X) = \begin{cases} 2 & 1 \leq p \leq 2 \\ p & 2 \leq p \end{cases} .$$

Once we have this definition, that is related to the geometry of the space, we can state the result we need.

Theorem 5 (Bombal–Pérez-García–Villanueva [6])**.** *If X has cotype q, then there is a constant $C_X > 0$ such that for each m-homogeneous polynomial $P : c_0 \to X$,*

$$\Big(\sum_{i_1,\dots,i_m} \| \overset{\vee}{P}(e_{i_1},\dots,e_{i_m})\|^q\Big)^{1/q} \leq C_X^m \sup_{z\in B_{c_0}} \|P(z)\|.$$

Coming from this result to the description of $S(X)$ is by no means trivial. It requires to translate $S(X)$ into a problem on absolute convergence of power series on infinitely many variables and with values on X and then study the set of points for which every such power series converges absolutely. This needs deep results on the theory of summing operators. This work was done by Defant, García, Maestre and Pérez-García in [9], where they prove that

$$S(X) = 1 - \frac{1}{q(X)}.$$

2.1. **Hardy spaces.** We have seen that, via Bohr's transform, power series in infinitely many variables and Dirichlet series are very much connected to each other. Given a Banach space X, a vector valued version of Bohr's transform can be defined in a natural way.

Holomorphic functions on B_{c_0} led to the space $\mathscr{H}_\infty$. Another natural 'source' of power series is to consider Fourier series, and this leads us to Hardy spaces, that define natural spaces of Dirichlet series. This was first considered by Bayart in [2] for scalar valued Dirichlet series. We introduce them directly in the vector valued case.

For the torus $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and the N-dimensional polytorus $\mathbb{T}^N = \prod_{k=1}^N \mathbb{T}$ the Hardy spaces $H_p(\mathbb{T})$ and $H_p(\mathbb{T}^N)$ are very well known spaces of functions that can be defined either as radial limits of holomorphic functions or as subspaces of the corresponding L_p space. In order to define Hardy spaces of X-valued functions on infinitely many variables we take the second point of view.

We denote by dw the normalized Lebesgue measure on the infinite dimensional polytorus $\mathbb{T}^\infty = \prod_{k=1}^\infty \mathbb{T}$, i.e. the countable product measure of the normalized Lebesgue measure on $\mathbb{T}$. For any multi-index $\alpha = (\alpha_1,\dots,\alpha_n,0,\dots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in $\mathbb{Z}$) the α-th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^\infty, X)$ is given by

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha} dw.$$

Then, given $1 \leq p < \infty$, the X-valued Hardy space on $\mathbb{T}^\infty$ is the subspace of $L_p(\mathbb{T}^\infty, X)$ defined as

$$H_p(\mathbb{T}^\infty, X) = \Big\{ f \in L_p(\mathbb{T}^\infty, X) \;\big|\; \hat{f}(\alpha) = 0, \ \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \Big\}. \tag{9}$$

Assigning to each $f \in H_p(\mathbb{T}^\infty, X)$ its unique formal power series $\sum_\alpha \hat{f}(\alpha) z^\alpha$ we may consider $H_p(\mathbb{T}^\infty, X)$ as a subspace of $\mathfrak{P}(X)$, the set of formal power series in X. Consider the X-valued Bohr transform $\mathfrak{B}_X$ given by:

$$\begin{array}{ccc}
\text{FORMAL POWER SERIES IN } X & & \text{DIRICHLET SERIES IN } X \\
\mathfrak{P}(X) & \xrightarrow{\qquad\qquad} & \mathfrak{D}(X) \\
\sum_\alpha c_\alpha z^\alpha & \xrightarrow{c_\alpha = a_{p^\alpha}} & \sum_n a_n \frac{1}{n^s} \\
\bigcup & & \\
H_p(\mathbb{T}^\infty, X) & &
\end{array}$$

Then the Hardy space $\mathscr{H}_p(X)$ of Dirichlet series in X is defined as the image of $H_p(\mathbb{T}^\infty, X)$ under the Bohr transform $\mathfrak{B}_X$. This vector space of Dirichlet series together with the norm

$$\|D\|_{\mathscr{H}_p(X)} = \|\mathfrak{B}_X^{-1}(D)\|_{H_p(\mathbb{T}^\infty,X)}$$

forms a Banach space. In other words, Bohr's transform $\mathfrak{B}_X$ gives the identification

$$\mathscr{H}_p(X) = H_p(\mathbb{T}^\infty, X)\,, 1 \leq p < \infty.$$

We remark that, for $p = \infty$, this identification also defines a Banach space $\mathscr{H}_\infty(X)$ which in the scalar case $X = \mathbb{C}$ coincides with the one given above. However, these two ways of defining $\mathscr{H}_\infty(X)$ are different for arbitrary X.

There is also a definition of $\mathscr{H}_p(X)$ which does not rely on Bohr's transform. For each finite Dirichlet polynomial $D = \sum_{k=1}^N a_k n^{-s}$, define its $\mathscr{H}_p(X)$-norm as:

$$\|D\|_{\mathscr{H}_p(X)} = \lim_{T\to\infty} \Big(\frac{1}{2T} \int_{-T}^{T} \Big\| \sum_{k=1}^{n} a_k \frac{1}{n^t} \Big\|_X^p dt \Big)^{1/p}.$$

By the Birkhoff–Khinchine ergodic theorem, the completion of the space of Dirichlet polynomials with this norm is our space $\mathscr{H}_p(X)$ (see e.g. Bayart [2] for the scalar case, the vector-valued case follows exactly the same way).

Motivated by (4) we define for $D \in \mathfrak{D}(X)$ and $1 \leq p < \infty$

$$\sigma_{\mathscr{H}_p(X)}(D) := \inf \Big\{ \sigma \in \mathbb{R} \mid \sum_n \frac{a_n}{n^\sigma} \frac{1}{n^s} \in \mathscr{H}_p(X) \Big\},$$

and motivated by (8) we consider

$$S_p(X) := \sup_{D \in \mathfrak{D}(X)} \sigma_a(D) - \sigma_{\mathscr{H}_p(X)}(D) = \sup_{D \in \mathscr{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). A result of Bayart [2] shows that for every $1 \leq p < \infty$

$$S_p(\mathbb{C}) = \frac{1}{2}. \tag{10}$$

As observed by Helson [14], this result is somehow surprising, since $\mathscr{H}_\infty(\mathbb{C})$ is much smaller than $\mathscr{H}_p(\mathbb{C})$.

Our aim is to give a vector valued counterpart of Bayart's result. As it happened in the previous cases, we need a sort of Bohnenblust–Hille type of inequality. It is the following result from [7]. Note that, since $\int_{\mathbb{T}^N} \|P(z)\| dz \leq \sup_{z \in B_{c_0}} \|P(z)\|$, it implies Theorem 5.

Theorem 6. *If X has cotype q, then there is a constant $C > 0$ such that, for each m-homogeneous polynomial $P : \mathbb{C}^N \to X$, and every $N \in \mathbb{N}$,*

$$\Big(\sum_{i_1,\ldots,i_m=1}^{N} \| \overset{\vee}{P}(e_{i_1}, \ldots, e_{i_m}) \|^q \Big)^{1/q} \leq C_X^m \int_{\mathbb{T}^N} \|P(z)\| dz.$$

Letting $N \to +\infty$, the integral on the right hand side goes to the norm of the polynomial P in the Hardy space $H_1(\mathbb{T}^\infty, X)$. With this, and identifying the space $H_1(\mathbb{T}^\infty, X)$ with the Hardy space of Dirichlet series $\mathscr{H}_1(X)$, it is possible to prove that $S_1(X) \leq 1 - \frac{1}{q(X)}$. From the natural containments between Hardy spaces, for $1 \leq p \leq \infty$ we have

$$1 - \frac{1}{q(X)} = S(X) \leq S_p(X) \leq S_1(X) \leq 1 - \frac{1}{q(X)}.$$

So we have obtained the following.

Theorem 7. *Let $q(X)$ be the optimal cotype of the Banach space X. Then, for $1 \le p \le \infty$ we have*

$$S_p(X) = 1 - \frac{1}{q(X)}.$$

This result has an immediate translation in terms of absolute convergence of power series. Indeed, let us define

$$M_p(X) = \sup\Big\{ r : \sum_\alpha \|c_\alpha\| |z|^\alpha < \infty \text{ for } z \in \ell_r \cap B_{c_0},\ f \in H_p(B_{c_0}, X) \Big\}.$$

Corollary 8. *For each Banach space X and $1 \le p \le \infty$ we have*

$$M_p(X) = \frac{q(X)}{q(X)-1}.$$

Remark 9. Let us finish this note by observing that, although by (10) the width of the strip of uniform but not absolute convergence is the same for all spaces $\mathscr{H}_p(\mathbb{C})$, the fact that $\mathscr{H}_\infty(\mathbb{C})$ has fewer Dirichlet series still makes a difference. Since $S_p(\mathbb{C})$ is defined as a supremum, (10) means that for every $\sum_n a_n \frac{1}{n^s} \in \mathscr{H}_p(\mathbb{C})$ we have

$$\sum_{n=1}^\infty |a_n| \frac{1}{n^{\frac{1}{2}+\varepsilon}} < \infty, \quad \forall \varepsilon > 0.$$

The question now is: *can we even get to $\varepsilon = 0$?* (we say in this case that the strip is attained). Here the difference between $\mathscr{H}_\infty(\mathbb{C})$ and other $\mathscr{H}_p(\mathbb{C})$ arises: Theorem 1.1 in [1] shows that in $\mathscr{H}_\infty(\mathbb{C})$ the strip is attained, whereas in $\mathscr{H}_p(\mathbb{C})$ for every $1 \le p < \infty$ it is not. An interesting question could be to study when $S_p(X)$ is attained in this sense.

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