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A REMARK ON QUANTUM MOMENTUM MAPS AND CLASSICAL ANOMALIES

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ABSTRACT. In order to answer the proposal given by P. Xu in [9], the existence of a quantum momentum map based on the existence of a classical momentum map is studied by O. Kravchenko in [4] and Müller-Bahns and Neumaier in [6]. In both papers only Hamiltonian actions are considered.

In these notes, we analyze the existence of a quantum momentum map based on the existence of a classical momentum map defined from a weakly Hamiltonian action. Some classical anomalies and quantum momentum maps are related.

1. INTRODUCTION

In the last years, many papers have considered the relation between classical and quantum symmetries in mechanical systems. The fundamental role that the momentum map plays in the analysis of classical mechanical systems with symmetries is well known. In the framework of deformation quantization, the quantum momentum map (q.m.m.) plays the role of a quantum analogous to the classical momentum map (c.m.m.).

In an interesting work ([9]), Xu has proved that a q.m.m. always recovers an Ad^*-equivariant c.m.m. He also raised the question whether the existence of a c.m.m. guarantees the existence of a q.m.m. that recovers it at the classical limit.

O. Kravchenko ([4]) proposed a definition of a q.m.m. which given rise a positive answer to the question posed by Xu. This definition allows consider cocycles that can be appear in the process of deformation quantization.

On the other hand, by considering a slightly different definition of q.m.m., Müller-Bahns and Neumaier ([6]) have given a negative answer to this question and have established necessary and sufficient conditions so that the existence of a c.m.m. implies the existence of a q.m.m. associated.

In these works only Hamiltonian actions, which define Ad^*-equivariant classical momentum maps, are considered. The aim of these notes is to adapt this kind of ideas in order to include weakly Hamiltonian actions which give rise to non Ad^*-equivariant classical momentum maps. A notion of a q.m.m. which allows us to recover a non Ad^*-equivariant c.m.m. at the classical limit is considered.

When a classical system is quantized can be appear anomalous terms usually called quantum anomaly. These anomalies are seen as a quantum effect. An example of this fact is to quantize conserved currents of a Lagrangian which close the algebra of classical observable but give rise to central terms on the commutator of quantum observable. These central terms are known as Schwinger terms and have an important physical meaning (see, for example, [8]).

The notion of classical anomaly has been introduced to describe a classical counterpart of a quantum anomaly ([7]). A classical anomaly occurs, for example, when a system admits a weakly Hamiltonian action ([4]). In this case, the Poisson bracket shows this anomaly which

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appears at classical level and it is conserved by quantizing. Our interest is to consider this kind of classical anomalies and its quantum counterpart in the framework of deformation quantization.

These notes are organized as follows. In section 2 the definitions of Ad^*-equivariant and non Ad^*-equivariant c.m.m. are recalled. In section 3 a very short review of deformation quantization is given. Also the definition of q.m.m. and some of the results proved in [6] and [4] are recalled.

In section 4 another notion of q.m.m. is considered. Given a non Ad^*-equivariant c.m.m., two different ways to consider it quantum counterpart are analyzed. Also the relation between quantum momentum maps and classical and quantum anomalies is considered.

2. Classical momentum maps

The momentum map plays a fundamental role in the analysis of classical mechanical systems with symmetries. In this section, we recall its definition and some of its important properties (for more details see, for example, [1] or [5]).

Let us consider a symplectic manifold (M,ω) and the linear space of differentiable functions on M with values in $\mathbb{R}$, $C^\infty(M)$. Let X_f be the Hamiltonian vector field of f defined by the condition $i_{X_f}\omega = d\,f$, where i_X is the contraction of the 2-form ω by X and d is the exterior differential operator on M. It is well known that $C^\infty(M)$ admits a canonical structure of Lie algebra associated to the form ω given by the Poisson bracket defined as follows. That is, $\{f,g\} = \omega(X_f,X_g)$ if f and $g \in C^\infty(M)$. The adjoint representation ad_∞ of $C^\infty(M)$ on itself is given by $(ad_\infty)_f(\cdot) = \{f,\,\cdot\,\}$.

We consider a symplectic left action of a Lie group G on M. That is, there exists a differential mapping $\phi : G\times M \to M$ such that $\phi_g^*\omega = \omega, \forall g\in G$, where ϕ_g^* is the pull-back of the diffeomorphism $\phi_g : M\to M$ given by $\phi_g(m) = \phi(g,m)$. If $\mathbf{g}$ is the Lie algebra of the group G, X_ξ denotes the infinitesimal generator associated to the action ϕ corresponding to $\xi\in\mathbf{g}$, and $\mathbf{g}^*$ denotes the dual space of $\mathbf{g}$. An action of G on M canonically induces a representation ρ of the Lie algebra $\mathbf{g}$ on $C^\infty(M)$ defined as $\rho(\xi)(f) = -L_{X_\xi} f$, where L denotes the Lie derivative. Under such action, $C^\infty(M)$ becomes a $\mathbf{g}$-module.

A differential function $J_0 : M\to\mathbf{g}^*$ is a *classical momentum map* for the action ϕ of G on M if $\langle J_0(m),\xi\rangle = \mathbf{J}_0(\xi)(m)$ for all $m\in M$ and $\xi\in\mathbf{g}$, where $\mathbf{J}_0(\xi)\in C^\infty(M)$ satisfies that $d\mathbf{J}_0(\xi) = i_{X_\xi}\omega$ for all $\xi\in\mathbf{g}$. Thus, a momentum map can be considered as an application $\mathbf{J}_0 : \mathbf{g}\to C^\infty(M)$ such that $X_{\mathbf{J}_0(\xi)} = X_\xi$ for all $\xi\in\mathbf{g}$. It is clear that $\{\mathbf{J}_0(\xi),f\} = \omega(X_\xi,X_f) = \rho(\xi)(f) = (ad_\infty)_{\mathbf{J}_0(\xi)}(f)\}$ for all $f\in C^\infty(M)$. A c.m.m. J_0 is Ad^**-equivariant* if $J_0(\phi_g(m)) = \mathrm{Ad}^*_{g^{-1}}J_0(m)$ for all $g\in G$, where Ad^* denotes the coadjoint action of G on $\mathbf{g}^*$.

In order to consider the cohomology of $\mathbf{g}$ with coefficients in $C^\infty(M)$ and other cohomologies derived from it, we recall the definition of the cohomology of $\mathbf{g}$ with coefficients in a $\mathbf{g}$-module V. Given a linear space V and ψ a representation of $\mathbf{g}$ in V, let us consider $C^k(\mathbf{g},V)$ the space of alternate k-multilineal maps α on $\mathbf{g}$ with values in V. The Chevalley–Eilenberg coboundary operator associated $\delta_\psi : C^k(\mathbf{g},V)\to C^{k+1}(\mathbf{g},V)$ for all $k\in\mathbb{N}$ is given by

$$\delta_\psi(\alpha)(\xi_1,\xi_2,\ldots,\xi_{k+1}) = \sum_{i=0}^{k+1}(-1)^{i+1}\psi(\xi_i)\left(\alpha(\xi_1,\xi_2,\ldots,\hat{\xi}_i,\ldots,\xi_{k+1})\right) + \sum_{i<j}\alpha\left([\xi_i,\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_{k+1}\right)$$

where the symbol ˆ means that the variable under it has been deleted.

An element $\alpha \in C^k(\mathbf{g},V)$ is a k-cocycle if $\delta_\psi(\alpha)=0$, and it is a k-coboundary if there exists an element $\beta \in C^{k-1}(\mathbf{g},V)$ such that $\delta_\psi(\beta)=\alpha$. If $Z^k_\psi(\mathbf{g},V)$ is the space of the k-cocycles and $B^k_\psi(\mathbf{g},V)$ is the space of the k-coboundaries, $H^k_\psi(\mathbf{g},V)=Z^k_\psi(\mathbf{g},V)\ /\ B^k_\psi(\mathbf{g},V)$ is the k-group of the cohomology of $\mathbf{g}$ with coefficients in V. It is well know that the extensions of $\mathbf{g}$ by V are characterized by the group $C^2(\mathbf{g},V)$.

A G-action on M is called Hamiltonian if there exists a c.m.m. $\mathbf{J}_0 \in C^1(\mathbf{g},C^\infty(M))$ such that is a Lie algebra homomorphism. That is, $\mathbf{J}_0([\xi,\eta])=\{\mathbf{J}_0(\xi),\mathbf{J}_0(\eta)\}$ for all $\xi,\eta\in\mathbf{g}$. A straightforward computation shows that if J_0 is Ad^*-equivariant then $\mathbf{J}_0$ is a Lie algebra homomorphism.

A G-action on M is called weakly Hamiltonian if there exists a c.m.m. $\mathbf{J}_0$ that is not a Lie algebra homomorphism. Given a weakly Hamiltonian G-action on M the application $\sigma: G\to\mathbb{R}$ given by $\sigma(g)=J_0(\phi_g(m))-\mathrm{Ad}^*_g(J_0(m))$ for some $m\in M$ measures the lack of Ad^*-equivariance of the momentum J_0. It is easy to check that this function is a 1-cocycle on G that takes values in $\mathbb{R}$. In a canonical way, this 1-cocycle gives rise to a 2-cocycle on $\mathbf{g}$ with values in $\mathbb{R}$, $\Sigma:\mathbf{g}\times\mathbf{g}\to\mathbb{R}$ given by $\Sigma(\xi,\eta)=\langle d\hat{\sigma}_\eta(e),\xi\rangle$, where $\hat{\sigma}_\eta: G\to\mathbb{R}$ is defined as $\hat{\sigma}_\eta(g)=\langle\sigma(g),\eta\rangle$. It is easy to see that Σ is a 2-cocycle on $\mathbf{g}$ with values in $\mathbb{R}$ associated to the trivial action of $\mathbf{g}$ on $\mathbb{R}$ and $\Sigma(\xi,\eta)=\{\mathbf{J}_0(\xi),\mathbf{J}_0(\eta)\}-\mathbf{J}_0([\xi,\eta])$. Thus, a non Ad^*-equivariant c.m.m. $\mathbf{J}_0$ canonically defines the Lie algebra $\widetilde{\mathbf{g}}=\mathbf{g}\oplus\mathbb{R}$, the central extension of $\mathbf{g}$ associated to the 2-cocycle Σ. Its Lie commutator is given by $[(\xi,a),(\eta,b)]=([\xi,\eta],\Sigma(\xi,\eta))$ for all $\xi,\eta\in\mathbf{g}$ and $a,b\in\mathbb{R}$. This central extension will be considered in Section 4.

3. Deformation quantization and quantum momentum maps

3.1. Deformation quantization. In this subsection we recall some basic aspects of deformation quantization ([2], [6], [9]).

Let us consider a symplectic manifold (M,ω) and $\{\cdot,\cdot\}$ the Lie algebra structure on $C^\infty(M)$ canonically associated to ω. Given the Plank constant $\hbar$, the set $C^\infty(M)[[\hbar]]$ is the vector space of formal power series in the parameter $\hbar$ with coefficients in $C^\infty(M)$. That is,

$$C^\infty(M)[[\hbar]]=\left\{\sum_{r=0}^{\infty} f_r\hbar^r\ :\ f_r\in C^\infty(M),\forall\, r\ge 0\right\}.$$

In general, for any vector space V,

$$V[[\hbar]]=\left\{\sum_{r=0}^{\infty} v_r\hbar^r\ :\ v_r\in V,\forall\, r\ge 0\right\}.$$

A *deformation quantization* of $C^\infty(M)$, or a *star product* $\star$, is an associative algebra structure on $C^\infty(M)[[\hbar]]$ of the form

$$f\star g=f\,g-\frac{i\hbar}{2}\{f,g\}+\sum_{r=2}^{\infty}C_r(f,g)\,\hbar^r,\quad \forall\, f \text{ and } g\in C^\infty(M)[[\hbar]],$$

where each $C_r(\cdot,\cdot)$ is a bidifferential operator verifies that $C_r(f,g)=(-1)^r C_r(g,f)$ and $C_r(1,f)=C_r(f,1)$, for all f and $g\in C^\infty(M)[[\hbar]]$.

A star product $\star$ on $C^\infty(M)[[\hbar]]$ gives it a Lie algebra structure by means of the bracket $[f,g]_\star=f\star g-g\star f$. The adjoint representation $(ad_\star)$ of $C^\infty(M)[[\hbar]]$ on itself is given by $(ad_\star)_\gamma(\cdot)=[\gamma,\cdot]_\star$. The existence proof of star products on a general symplectic manifold was first obtained by Wilde and Lecomte [3] using a homological argument.

We will consider, as in [6], a generalized Fedosov's $\star$ product on a symplectic manifold (M,ω) stemming from a Weyl product $\circ$ and a triad (∇,Ω,s) constituted by a flat torsion free symplectic connection ∇ on M, a 2-form formal series Ω on M, and a certain formal series s of symmetric contravariant tensor fields on M without terms of symmetric degree 1.

Given a Lie group G that symplectically acts in (M,ω), the action ρ of $\mathbf{g}$ on $C^\infty(M)$ defined in section 2 can be naturally extended to an action ρ_c of $\mathbf{g}$ on $C^\infty(M)[[\hbar]]$ in the following way:

$$\rho_c(\xi)\left(\sum_{r\geq 0} f_r\hbar^r\right) = -\sum_{r\geq 0}(L_{X_\xi} f_r)\,\hbar^r.$$

Thus, $C^\infty(M)[[\hbar]]$ turns in a $\mathbf{g}$-module. We shall consider the cohomology of $\mathbf{g}$ with coefficients in $C^\infty(M)[[\hbar]]$ and its associated coboundary operator δ_{ρ_c}.

A star product $\star$ is called $\mathbf{g}$-invariant if $\rho_c(\xi)(f\star g) = (\rho_c(\xi)(f)\star g) + (f\star \rho_c(\xi)(g))$, for all $f,g\in C^\infty(M)[[\hbar]]$ and for all $\xi\in\mathbf{g}$. In the next sections these star products will be considered.

3.2. **Quantum momentum maps.** In this subsection we recall the definitions of q.m.m. considered in [6] and [4]. In addition, we present some results proven in these papers.

Definition 1. *Let us consider a Hamiltonian G-action on M and a $\mathbf{g}$-invariant star product $\star$. According to* [6], $\mathbf{J}\in C^1(\mathbf{g},C^\infty(M))[[\hbar]]$ *is a* quantum Hamiltonian *for the action ρ_c if*

$$\rho_c(\xi)(\cdot) = \frac{1}{\hbar}(ad_\star)_{\mathbf{J}(\xi)}(\cdot) = \frac{1}{\hbar}[\mathbf{J}(\xi),\,\cdot\,]_\star, \quad \forall\,\xi\in\mathbf{g}. \tag{1}$$

A quantum Hamiltonian $\mathbf{J}$ *is a* q.m.m. *if* $\mathbf{J}:\mathbf{g}\longrightarrow (C^\infty(M)[[\hbar]],\frac{1}{\hbar}[\cdot,\cdot]_\star)$ *is a Lie algebra homomorphism. That is,*

$$\frac{1}{\hbar}(\mathbf{J}(\xi)\star\mathbf{J}(\eta)-\mathbf{J}(\eta)\star\mathbf{J}(\xi)) = \mathbf{J}([\xi,\eta]), \quad \forall\,\xi \text{ and } \eta\in\mathbf{g}. \tag{2}$$

Note 1. *Notice that* $\mathbf{J}$ *can be written as* $\mathbf{J}=\mathbf{J}_0+\mathbf{J}_+$, *where* $\mathbf{J}_0\in C^1(\mathbf{g},C^\infty(M))$ *and* $\mathbf{J}_+\in \hbar C^1(\mathbf{g},C^\infty(M))[[\hbar]]$. *It is very simple to see that the zeroth order in $\hbar$ of (1) is equivalent to $\mathbf{J}_0$ being a c.m.m. for the action ρ, and that the zeroth order in $\hbar$ of (2) just means* Ad**-equivariance of this c.m.m. Thus, it is clear that a q.m.m. always gives rise to an* Ad**-equivariant c.m.m.*

A $\mathbf{g}$-invariant Fedosov star product $\star$ for (M,ω) obtained from (∇,Ω,s) admits a quantum Hamiltonian if and only if there is an element $\mathbf{J}\in C^1(\mathbf{g},C^\infty(M))[[\hbar]]$ such that $d\mathbf{J}(\xi) = i_{X_\xi}(\omega+\Omega)$ for all $\xi\in\mathbf{g}$.

If $\mathbf{J}$ is a quantum Hamiltonian for the a $\mathbf{g}$-invariant Fedosov star product $\star$, then $\lambda\in C^2(\mathbf{g},C^\infty(M))[[\hbar]]$, defined by

$$\lambda(\xi,\eta) = \frac{1}{\hbar}(\mathbf{J}(\xi)\star\mathbf{J}(\eta)-\mathbf{J}(\eta)\star\mathbf{J}(\xi)) - \mathbf{J}([\xi,\eta]),$$

is an element of $Z^2(\mathbf{g},\mathbb{R})[[\hbar]]$ explicitly given by $\lambda(\xi,\eta) = (\omega+\Omega)(X_\xi,X_\eta)-\mathbf{J}([\xi,\eta])$.

Given $\mathbf{J}_0$ an Ad*-equivariant c.m.m., there exists $\mathbf{J}\in(\mathbf{g},C^\infty(M))[[\hbar]]$ a q.m.m. that recovers $\mathbf{J}_0$ if and only if there exists $\mathbf{J}_+\in\hbar.C^1(\mathbf{g},C^\infty(M))[[\hbar]]$ such that $i_{X_\xi}\Omega = d\,\mathbf{J}_+(\xi)$ and $\Omega(X_\xi,X_\eta) = (\delta_{\rho_c}\mathbf{J}_+)(\xi,\eta)$ for all $\xi,\eta\in g$.

Let us notice that this equivalence gives rise, in general, a negative answer to the question given by Xu, and this answer is positive if and only if the 2-cocycle $\Omega(X_\xi,X_\eta)$ is cohomologically trivial.

On the other hand, in [4] O. Kravchenko proposed a slightly different definition of q.m.m. by considering the projective representation of $\mathbf{g}$ on $C^\infty(M)[[\hbar]]$ associated to the 2-cocycle λ. A q.m.m. is a Lie algebra map $\mu^{\mathrm{Lie}}:\mathbf{g}\to \mathrm{Inn}\,C^\infty(M)[[\hbar]]$, where $\mathrm{Inn}\,C^\infty(M)[[\hbar]]$

is the inner automorphisms of $C^\infty(M)[[\hbar]]$, which inherit the Lie algebra structure from $C^\infty(M)[[\hbar]]$. In particular, $\lim_{\hbar\to 0}\mu^{\mathrm{Lie}}(\xi)(f)=\{J(\xi),f\}$ for all $\xi\in\mathbf{g}$ and $f\in C^\infty(M)$.

Thus, if there exists $\mathbf{J}_+\in\hbar.C^1(\mathbf{g},C^\infty(M))[[\hbar]]$ such that $i_{X_\xi}\Omega=d\mathbf{J}_+(\xi)$ for all $\xi\in g$, there exists a q.m.m. μ^{Lie} given by $\mu^{\mathrm{Lie}}(\xi)(f)=[\mu(\xi),f]_*$, where $\mu=\mathbf{J}_0+\mathbf{J}_+$. This fact gives rise to a positive answer to the question posed in [9].

Note 2. *It is clear that according to* [6]*, there exists a q.m.m. from a c.m.m. if and only if deformation quantization does not give rise to an anomaly; that is, the cocycle λ is cohomologically trivial. According to* [4] *there exists a q.m.m. from a c.m.m. even if deformation quantization gives rise to anomalous terms. It is clear that classical anomalies are not considered in these works.*

4. Quantum counterpart of a non Ad*-equivariant classical momentum maps

There are many classical mechanical systems with symmetries that give rise to classical anomalies. A simple example is $\mathbb{R}^n$ acting by translations on $\mathbb{R}^n$ with the canonical symplectic structure. Another non Ad*-equivariant momentum map appear if we consider lift cotangent actions with symplectic forms are canonical modified by the addition of a magnetic term.

Also a conserved current of a Lagrangian can be seen as a non Ad*-equivariant c.m.m. that gives rise to a Lie algebra central extension. The affine Kac–Moody algebra that appears when compute the equal time commutator for a 2-dimensional theory, is a central extension of a loop algebra. The Virasoro algebra, that plays a very important role in 2-dimensional conformal theories, is a central extension of the Lie algebra of the group of diffeomorphisms of S^1. In both cases, the classical anomaly is given by a non Ad*-equivariant c.m.m.

In order to treat the quantum counterpart of a classical anomaly defined from a non Ad*-equivariant classical momentum map we need to define the quantum counterpart of a non Ad*-equivariant c.m.m.

4.1. Ad*-equivariant and anomalous quantum momentum maps.

As we said in Note 1, the condition (1) that defines a quantum Hamiltonian on Definition 1 describes the characterization of the quantum counterpart of a c.m.m. Meanwhile, the condition (2) that defines a quantum momentum map on the same definition, corresponds to the property of Ad*-equivariance of a c.m.m. For this simple reason, analogously to the classical case, we propose that a quantum Hamiltonian should be called q.m.m. and a quantum momentum map should be called Ad*-equivariant q.m.m.

In this framework, the following definition generalizes Definition 1.

Definition 2. *Given a* $\mathbf{g}$*-invariant star product* $\star$*,* $\mathbf{J}\in C^1(\mathbf{g},C^\infty(M))[[\hbar]]$ *is a q.m.m. of the action* ρ_c *if*

$$\rho_c(\xi)(\cdot)=\frac{1}{\hbar}(ad_\star)_{\mathbf{J}(\xi)}(\cdot)=\frac{1}{\hbar}[\mathbf{J}(\xi),\,\cdot\,]_\star,\quad \forall\,\xi\in\mathbf{g}. \tag{3}$$

If $\mathbf{J}$ *is not a Lie algebra homomorphism,* $\mathbf{J}$ *is called non* Ad**-equivariant or anomalous q.m.m.*

Let us notice that a non Ad*-equivariant c.m.m. can not be considered as the classical limit of an Ad*-equivariant q.m.m. but can be recovered from an anomalous q.m.m.

Given a non Ad*-equivariant c.m.m., according to the proven results in [6] it is immediate to characterize the existence of an anomalous q.m.m. that recovers $\mathbf{J}_0$.

Proposition 1. *Given a non* Ad^**-equivariant c.m.m.* $\mathbf{J}_0$*, there exists* $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_+$ *an anomalous q.m.m. if and only if there exists* $\mathbf{J}_+ \in \hbar C^1(\mathbf{g}, C^\infty(M))[[\hbar]]$ *such that* $i_{X_\xi}\Omega = d\,\mathbf{J}_+(\xi)$*.*

In this way, we arrive at the same conclusion that O. Kravkencho by considering non necessarily Ad^*-equivariant classical momentum maps.

4.2. Ad^***-equivariant quantum momentum map associated to the canonically extended classical momentum map.** As we recall in Section 2, in the case of a weakly Hamiltonian action with a non Ad^*-equivariant momentum map $\mathbf{J}_0$, there exists a canonical central extension of the Lie algebra $\mathbf{g}$ given by the 2-cocycle Σ that measures the non Ad^*-equivariance of the momentum map.

We assume that the extension $\widetilde{\mathbf{g}}$ of the Lie algebra $\mathbf{g}$ corresponds to an extension $\widetilde{G}$ of the Lie group G.

Let us consider the trivially extended action of $\widetilde{G}$ on M, $\widetilde{\phi} : \widetilde{G} \times M \to M$ defined as $\widetilde{\phi}_{(g,A)}(m) = \phi_g(m)$ for all $(g,A) \in \widetilde{G}$ and $m \in M$. Then the infinitesimal generator $\widetilde{X}_{(\xi,a)}$ associated to $(\xi,a) \in \widetilde{\mathbf{g}}$ coincides with the infinitesimal generator X_ξ for all $\xi \in \mathbf{g}$. In a canonical way, we can define a representation $\widetilde{\rho} : \widetilde{\mathbf{g}} \times C^\infty(M) \to C^\infty(M)$ given by $\widetilde{\rho}(\xi,a)(f) = -L_{\widetilde{X}_{(\xi,a)}} f$, and its associated coboundary operator $\delta_{\widetilde{\rho}}$.

Given a non Ad^*-equivariant c.m.m. $\mathbf{J}_0$, we can define a momentum map that results equivariant with respect to an extended coadjoint action of $\widetilde{\mathbf{g}}^*$ defined for all $(\alpha,x) \in \widetilde{\mathbf{g}}^*$ as $\widetilde{\mathrm{Ad}}^*_{(g,A)}(\alpha,x) = (\mathrm{Ad}^*_g \alpha + \sigma(g), x)$.

The application $\widetilde{\mathbf{J}}_0 : \widetilde{\mathbf{g}} \to C^\infty(M)$ defined as $\widetilde{\mathbf{J}}_0(\xi,a)(m) = \mathbf{J}_0(\xi)(m) + a$ is called canonically extended c.m.m. associated to the action $\widetilde{\phi}$ of $\widetilde{G}$ on M.

Now, we will study the existence of an Ad^*-equivariant q.m.m. associated to $\widetilde{\mathbf{J}}_0$. In the first place, let us notice that the representation $\widetilde{\rho}$ of $\widetilde{\mathbf{g}}$ on $C^\infty(M)$ can be canonically extended to the space $C^\infty(M)[[\hbar]]$ as follows. Let $\widetilde{\rho}_c : \widetilde{\mathbf{g}} \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ be given by $\widetilde{\rho}_c(\xi,a)(\sum_{r\geq 0} f_r\hbar^r) = \sum_{r\geq 0} \widetilde{\rho}(\xi,a)(f_r)\hbar^r = \sum_{r\geq 0} \rho(\xi)(f_r)\hbar^r$, and $\delta_{\widetilde{\rho}_c}$ its associated coboundary operator. It is easy to verify that is equivalent to consider $\mathbf{g}$-invariant or $\widetilde{\mathbf{g}}$-invariant Fedosov star products.

Lemma 1. *A Fedosov star product* $\star$ *is* $\widetilde{\mathbf{g}}$*-invariant if and only if it is a* $\mathbf{g}$*-invariant Fedosov star product.*

Proof. A Fedosov star product $\star$ is $\widetilde{\mathbf{g}}$-invariant if and only if

$$\widetilde{\rho}_c(\xi,a)(f \star g) = (\widetilde{\rho}_c(\xi,a)(f)) \star g + f \star (\widetilde{\rho}_c(\xi,a)(g)).$$

Since $\widetilde{\rho}_c(\xi,a)(f) = -L_{\widetilde{X}_{(\xi,a)}} f = -L_{X_\xi} f = \rho_c(\xi)(f)$, then the above condition is equivalent to $\rho(\xi)(f \star g) = (\rho(\xi)(f) \star g) + (f \star \rho(\xi)(g))$. That is, $\star$ is $\mathbf{g}$-invariant. ■

Now, let us consider a $\mathbf{g}$-invariant Fedosov star product $\star$ defined from (∇, Ω, s) as has been described in Section 3. According to [4] we will suppose that there exists $\widetilde{\mathbf{J}}_+ \in \hbar\, C^1(\widetilde{\mathbf{g}}, C^\infty(M))[[\hbar]]$ such that $i_{\widetilde{X}_{(\xi,a)}}\Omega = d\widetilde{\mathbf{J}}_+(\xi,a)$. Then, we can consider the q.m.m. $\widetilde{\mu}^{\mathrm{Lie}} : \widetilde{\mathbf{g}} \to \mathrm{Inn}\, C^\infty(M)[[\hbar]]$ given by $\widetilde{\mu}^{\mathrm{Lie}}((\xi,a), f) = [\widetilde{\mu}((\xi,a)), f]_*$, where $\widetilde{\mu} = \widetilde{\mathbf{J}}_0 + \widetilde{\mathbf{J}}_+$.

A direct computation shows that $\mathbf{J}_0$ can be recovered as the classical limit of the restriction of $\widetilde{\mu}^{\mathrm{Lie}}$ to $\mathbf{g}$.

Proposition 2. $\breve{\mu} = \widetilde{\mu}|_{\mathbf{g}\oplus\{0\}}$ *is an anomalous q.m.m. that recovers the non* Ad^**-equivariant c.m.m.* $\mathbf{J}_0$*.*

Proof. It is clear that $\breve{\mu}$ is a q.m.m., because

$$\rho_c(\xi)(\cdot) = \widetilde{\rho}_c(\xi,0)(\cdot) = \frac{1}{\hbar}[\widetilde{\mu}(\xi,0),\ \cdot\ ]_\star = \frac{1}{\hbar}[\breve{\mu}(\xi),\ \cdot\ ]_\star, \qquad \text{for all } \xi \in \mathbf{g}.$$

Also we can see that $\breve{\mu}$ is not a Lie algebra homomorphism,

$$[\breve{\mu}(\xi),\breve{\mu}(\eta)]_\star = [\widetilde{\mu}(\xi,0),\widetilde{\mu}(\eta,0)]_\star = \widetilde{\mu}([(\xi,0),(\eta,0)]) = \widetilde{\mu}([(\xi,\eta)],\Sigma(\xi,\eta)),$$

and $\breve{\mu}([\xi,\eta]) = \widetilde{\mu}([\xi,\eta],0)$. Then $\breve{\mu}$ is an anomalous q.m.m.

On the other hand, $\breve{\mu}(\xi) = \widetilde{\mu}(\xi,0) = \widetilde{\mathbf{J}}_0(\xi,0) + \widetilde{\mathbf{J}}_+(\xi,0) = \mathbf{J}_0(\xi) + \widetilde{\mathbf{J}}_+(\xi,0)$. Then it is clear that $\breve{\mu}$ recovers $\mathbf{J}_0$ at the classical limit. ■

Note 3. *By considering the definition of q.m.m. given in* [6]*, and assuming the existence of* $\widetilde{\mathbf{J}}_+ \in \hbar\, C^1(\widetilde{\mathbf{g}}, C^\infty(M))[[\hbar]]$ *such that* $i_{\widetilde{X}_{(\xi,a)}}\Omega = d\widetilde{\mathbf{J}}_+(\xi,a)$*, there exists an* Ad**-equivariant q.m.m. for* $\widetilde{\rho}_c$ *given by* $\widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}_0 + \widetilde{\mathbf{J}}_+ : \widetilde{\mathbf{g}} \to C^\infty(M)[[\hbar]]$ *if and only if*

$$\Omega\left(\widetilde{X}_{(\xi,a)},\widetilde{X}_{(\eta,b)}\right) = \left(\delta_{\widetilde{\rho}_c}\widetilde{\mathbf{J}}_+\right)_{((\xi,a),(\eta,b))}.$$

4.3. **Canonical extended anomalous quantum momentum map.** Given $\mathbf{J}_0$ a non Ad*-equivariant c.m.m. let us assume that there exists $\mathbf{J}_+$ such that $\mathbf{J}_+ \in \hbar C^1(\mathbf{g}, C^\infty(M))[[\hbar]]$ and $i_{X_\xi}\Omega = d\mathbf{J}_+(\xi)$.

We analyze the 2-cocycle $\lambda : \mathbf{g}\times\mathbf{g} \to \mathbb{R}[[\hbar]]$ given by $\lambda(\xi,\eta) = \frac{1}{\hbar}[\mathbf{J}(\xi),\mathbf{J}(\eta)]_\star - \mathbf{J}([\xi,\eta])$, where $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_+$.

It is clear that $\mathbb{R}[[\hbar]]$ becomes a $\mathbf{g}$-module by the trivial action. Thus, we can consider the central extension $\widehat{\mathbf{g}}$ of $\mathbf{g}$ by $\mathbb{R}[[\hbar]]$ associated to the 2-cocycle λ.

Thus, $\widehat{\mathbf{g}} = \mathbf{g} \oplus \mathbb{R}[[\hbar]]$ and $[(\xi,\sum_{r\geq 0} x_r\hbar^r),(\eta,\sum_{r\geq 0} y_r\hbar^r)] = ([\xi,\eta],\lambda(\xi,\eta))$, where $\xi,\eta \in \mathbf{g}$ and $\sum_{r\geq 0} x_r\hbar^r, \sum_{r\geq 0} y_r\hbar^r \in \mathbb{R}[[\hbar]]$. The Lie algebra $\widehat{\mathbf{g}}$ acts trivially in the extension component on $C^\infty(M)$ and this action is canonically extended on $C^\infty(M)[[\hbar]]$. That is, $\widehat{\rho}_c : \widehat{\mathbf{g}} \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ is given by $\widehat{\rho}_c(\xi,\sum_{r\geq 0} x_r\hbar^r)(\sum_{r\geq 0} f_r\hbar^r) = -\sum_{r\geq 0}(L_{X_\xi} f_r)\hbar^r$.

We will define an Ad*-equivariant quantum momentum map that canonically extends $\mathbf{J}$. Let us consider the application $\widehat{\mathbf{J}} : \widehat{\mathbf{g}} \to C^\infty(M)[[\hbar]]$ given by

$$\widehat{\mathbf{J}}(\xi,\sum_{r\geq 0} x_r\hbar^r) = \mathbf{J}(\xi) + \sum_{r\geq 0} x_r\hbar^r.$$

In the first place, $\widehat{\mathbf{J}}$ is a q.m.m. for the quantum action $\widehat{\rho}_c$,

$$\widehat{\rho}_c(\xi,\sum_{r\geq 0} x_r\hbar^r)(\cdot) = \frac{1}{\hbar}[\widehat{\mathbf{J}}(\xi),\ \cdot\ ]_\star, \quad \forall\, (\xi,\sum_{r\geq 0} x_r\hbar^r) \in \widehat{\mathbf{g}}.$$

In order to see that $\widehat{\mathbf{J}}$ is a Lie algebra homomorphism we will recall the following basic property.

Lemma 2. *If* $f \in \mathbb{R}[[\hbar]]$ *then* $[f,g]_\star = 0$*, for all* $g \in C^\infty(M)[[\hbar]]$.

Proof. Let $f = \sum_{r\geq 0} a_r\hbar^r \in \mathbb{R}[[\hbar]]$ and $g = \sum_{r\geq 0} g_r\hbar^r \in C^\infty(M)[[\hbar]]$, with $a_r \in \mathbb{R}$ and $g_r \in C^\infty(M)$ for all $r \geq 0$. Then,

$$[f,g]_\star = \left[\sum_{r\geq 0} a_r\hbar^r, \sum_{r\geq 0} g_r\hbar^r\right]_\star = \sum_{r\geq 0}(\sum_{i+j=r}[a_i,g_i]_\star)\hbar^r = 0,$$

where the last equality is fulfilled because $[a,g]_\star = 0$ for all $a \in \mathbb{R}$ and $g \in C^\infty(M)$. ■

Proposition 3. $\widehat{\mathbf{J}} : \widehat{\mathbf{g}} \longrightarrow (C^\infty(M)[[\hbar]], \frac{1}{\hbar}[\cdot,\cdot]_\star)$ *is a Lie algebra homomorphism.*

Proof. By definition of $\widehat{\mathbf{J}}$ and the bracket in $\widehat{\mathbf{g}}$,

$$\widehat{\mathbf{J}}([(\xi, \sum_{r\geq 0} x_r\hbar^r), (\eta, \sum_{r\geq 0} y_r\hbar^r)]) = \widehat{\mathbf{J}}([\xi,\eta], \lambda(\xi,\eta)) = \mathbf{J}([\xi,\eta]) + \lambda(\xi,\eta).$$

On the other hand,

$$\begin{aligned}\frac{1}{\hbar}[\widehat{\mathbf{J}}(\xi, \sum_{r\geq 0} x_r\hbar^r), \widehat{\mathbf{J}}(\eta, \sum_{r\geq 0} y_r\hbar^r)]_\star &= \frac{1}{\hbar}[\mathbf{J}(\xi) + \sum_{r\geq 0} x_r\hbar^r, \mathbf{J}(\eta) + \sum_{r\geq 0} y_r\hbar^r]_\star \\ &= \frac{1}{\hbar}[\mathbf{J}(\xi), \mathbf{J}(\eta)]_\star + \frac{1}{\hbar}[\mathbf{J}(\xi), \sum_{r\geq 0} y_r\hbar^r]_\star \\ &\quad + \frac{1}{\hbar}[\sum_{r\geq 0} x_r\hbar^r, \mathbf{J}(\eta)]_\star + \frac{1}{\hbar}[\sum_{r\geq 0} x_r\hbar^r, \sum_{r\geq 0} y_r\hbar^r]_\star \\ &= \frac{1}{\hbar}[\mathbf{J}(\xi), \mathbf{J}(\eta)]_\star.\end{aligned}$$

This last equality is fulfilled because the three last brackets are equal to zero. Then, by definition of λ, it is clear that $\widehat{\mathbf{J}}$ is a Lie algebra homomorphism. ■

Corollary 1. *The quantum momentum map $\widehat{\mathbf{J}}$, called canonical extended q.m.m. associated to $\mathbf{J}$, is an* Ad^**-equivariant q.m.m. corresponding to the quantum action $\widehat{\rho}_c$.*

Proposition 4. *The* Ad^**-equivariant q.m.m. $\widehat{\mathbf{J}}$ recovers the* Ad^**-equivariant c.m.m. $\widetilde{\mathbf{J}}_0$ at the classical limit.*

Proof. The application $\widehat{\mathbf{J}}$ can be written as

$$\widehat{\mathbf{J}}(\xi, \sum_{r\geq 0} x_r\hbar^r) = \mathbf{J}(\xi) + \sum_{r\geq 0} x_r\hbar^r = \mathbf{J}_0(\xi) + \mathbf{J}_+(\xi) + \sum_{r\geq 0} x_r\hbar^r.$$

So, the classical limit of $\widehat{\mathbf{J}}(\xi, \sum_{r\geq 0} x_r\hbar^r)$ agree with $\widetilde{\mathbf{J}}(\xi, x_0) = \mathbf{J}_0(\xi) + x_0$ for all $\xi \in \mathbf{g}$ and $x_0 \in \mathbb{R}$.

On the other hand, it is clear that $\widehat{\mathbf{g}}/_{[[\hbar]]} \sim \mathbf{g} \oplus \mathbb{R} = \widetilde{\mathbf{g}}$, and $C^\infty(M)[[\hbar]]/_{[[\hbar]]} \sim C^\infty(M)$. Thus we can consider that the Ad^*-equivariant q.m.m. $\widehat{\mathbf{J}}$ recovers the Ad^*-equivariant c.m.m. $\widetilde{\mathbf{J}}_0$ at the classical limit. ■

It is easy to see that there exists $\tilde{\mu}^{\mathrm{Lie}}$ if and only if there exists $\widehat{\mathbf{J}}$.

Lemma 3. *There exists $\widetilde{\mathbf{J}}_+ \in \hbar\, C^1(\widetilde{\mathbf{g}}, C^\infty(M))[[\hbar]]$ such that $i_{\widetilde{X}_{(\xi,a)}}\Omega = d\widetilde{\mathbf{J}}_+(\xi,a)$ if and only if there exists $\mathbf{J}_+ \in \hbar\, C^1(\mathbf{g}, C^\infty(M))[[\hbar]]$ such that $i_{X_\xi}\Omega = d\mathbf{J}_+(\xi)$. That is, there exists $\tilde{\mu}^{\mathrm{Lie}}$ if and only if there exists $\widehat{\mathbf{J}}$.*

Proof. If there exists $\widetilde{\mathbf{J}}_+$ that satisfies $i_{\widetilde{X}_{(\xi,a)}}\Omega = d\widetilde{\mathbf{J}}_+(\xi,a)$, it is clear that $\mathbf{J}_+ : \mathbf{g} \to C^\infty(M)[[\hbar]]$, defined as $\mathbf{J}_+(\xi) = \widetilde{\mathbf{J}}_+(\xi,0)$, satisfies $i_{\widetilde{X}_\xi}\Omega = d\widetilde{\mathbf{J}}_+(\xi)$.

Reciprocally, if there exists $\mathbf{J}_+ \in C^\infty(M)[[\hbar]]$ such that verifies $i_{\widetilde{X}_\xi}\Omega = d\widetilde{\mathbf{J}}_+(\xi)$, it is clear that $\widetilde{\mathbf{J}}_+ : \widetilde{\mathbf{g}} \to C^\infty(M)[[\hbar]]$, defined as $\widetilde{\mathbf{J}}_+(\xi,a) = \mathbf{J}_+(\xi)$, satisfies $i_{\widetilde{X}_{(\xi,a)}}\Omega = d\widetilde{\mathbf{J}}_+(\xi,a)$. ■

Note 4. *Notice that the quantum anomaly λ can be written as $\lambda(\xi,\eta) = \sum_{r\geq 0} \lambda_r(\xi,\eta)\hbar^r$, where $\lambda_r \in C^2(\mathbf{g},\mathbb{R})$ for all $r \in \mathbb{R}$, and its constant term $\lambda_0(\xi,\eta)$ is the classical anomaly $\lambda_0(\xi,\eta) = \{\mathbf{J}_0(\xi),\mathbf{J}_0(\eta)\} - \mathbf{J}_0([\xi,\eta]) = \Sigma(\xi,\eta)$. Then, the quantum anomaly λ*

- *contains only higher order terms if the G-action on M is Hamiltonian,*
- *includes the classical anomaly as its zero order term if the G-action on M is weakly Hamiltonian.*

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