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MONADIC SYMMETRIC BOOLEAN ALGEBRAS*

by

Manuel Abad and Luiz Monteiro

ABSTRACT. The purpose of this paper is to introduce and investigate a new (equational) class of algebras, which we call Symmetric Monadic Boolean Algebras or simply Algebras, as a system $(B,\exists,T)$ where B is a Boolean algebra, $(B,\exists)$ is a Monadic Boolean algebra [P.Halmos, 1962], (B,T) is a Symmetric Boolean algebra [Gr.C.Moisil, 1954,1972; A.Monteiro, 1966,1969], and $\exists Tx = T\exists x$, for all $x \in B$.

In §2 we characterize simple algebras and prove a decomposition theorem. We also obtain the number of automorphisms in the finite simple algebras.

In §3 we prove that the variety of Symmetric Monadic Boolean algebras is locally finite. As an application of these results we obtain in §4 the structure of the Symmetric Monadic Boolean algebra with a finite set of free generators.

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Received, September 1988.

1. INTRODUCTION

The object of this section is to give definitions and some well-known results of the theory of Monadic Boolean Algebras and Symmetric Boolean Algebras, that are relevant to this paper. For further information and references on these algebras the reader is referred to [7,8,9,10,11].

A *Monadic Boolean Algebra* is an algebra $(B,\exists)$ such that B is a Boolean algebra and $\exists$ is a unary operation (called quantification) defined on B satisfying the identities:

M1) $\exists 0 = 0$; M2) $x = x \wedge \exists x$; M3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$, [7]. For the sake of simplicity we say that B is a monadic algebra.

It is well known that if $K(B) = \{x \in B: \exists x = x\}$, then $K(B)$ is a Boolean subalgebra of B which is upper conditionally complete, that is, if $x \in B$ the family $\{k \in K: x \leqslant k\}$ has an infimum in K.

If B is a finite Boolean algebra with $|B| > 1$ and $A(B)$ is the set of all atoms of B, then there exists a one-to-one correspondence between the set of partitions of $A(B)$ and the set of quantification operations defined on B.

If $\{C_i\}_{1\leq i\leq t}$ is a partition of $A(B)$, then the elements $b_i = \bigvee_{x\varepsilon C_i} x$, $1 \leqslant i \leqslant t$, are the atoms of $K(B)$.

If $x \in B$ and $A(x) = \{a \in A(B): a \leqslant x\}$, then $x = \bigvee\{a: a \in A(x)\}$.

A *Symmetric Boolean Algebra* is an algebra (B,T) where B is a Boolean algebra and T is a Boolean automorphism of B such that $TTx = x$, for all $x \in B$ [8,9,10,11,2,3]. We say that B is a symmetric algebra.

1.1. LEMMA. If B is a symmetric algebra, $\{a_j\}_{j\varepsilon J}$ a family of elements in B and there exists $a = \bigvee_{j\varepsilon J} a_j$, then there exists $\bigvee_{j\varepsilon J} Ta_j$ and $T(\bigvee_{j\varepsilon J} a_j) = \bigvee_{j\varepsilon J} Ta_j$.

PROOF. We have $a_j \leqslant a$ for every $j \in J$, then $Ta_j \leqslant Ta$ for every $j \in J$. On the other hand, if $b \in B$ is such that $Ta_j \leqslant b$ for every $j \in J$, then $a_j = TTa_j \leqslant Tb$, and $a = \bigvee_{j\varepsilon J} a_j \leqslant Tb$. Therefore $Ta \leqslant TTb = b$. ∎

Let X be a non-empty set, B a symmetric algebra, and B^X the set of all functions from X into B. We can define on B^X the following pointwise operations:

$$(f \vee g)(x) = f(x) \vee g(x) \quad ; \quad (f \wedge g)(x) = f(x) \wedge g(x)$$
$$(-f)(x) = -(f(x)) \quad ; \quad (Tf)(x) = Tf(x)$$
$$0(x) = 0 \quad ; \quad 1(x) = 1$$

for all $x \in X$, and so $(B^X, \vee, \wedge, -, T, 0, 1)$ is a symmetric algebra.

If $f \in B^X$ and there exists the supremum (infimum) of the set $R(f) = \{f(x) : x \in X\}$, we note $\exists f = \bigvee R(f)$ $(\forall f = \bigwedge R(f))$. Then we have the following functions in B^X:

$$(\exists f)(x) = \exists f \quad ; \quad (\forall f)(x) = \forall f \quad , \quad x \in X.$$

1.2. DEFINITION. A symmetric subalgebra S of a symmetric Boolean algebra B^X is said to be a *monadic functional symmetric Boolean algebra* if it verifies:

S1) If $f \in S$, there exist the elements $\exists f$, $\forall f$ in B.

S2) If $f \in S$, $\exists f$ and $\forall f$ are elements of S.

1.3. LEMMA. If S is a monadic functional symmetric Boolean algebra, then the operation $\exists$ has the following properties, for $f,g \in S$:

E0) $\exists 0 = 0$; E1) $f \leqslant \exists f$; E2) $\exists(f \wedge \exists g) = \exists f \wedge \exists g$; E3) $\exists Tf = T\exists f$.

PROOF. The properties E0, E1 and E2 are well known (see for example [12])
E3) From Lemma 1.1 $T(\bigvee_{x \in X} f(x)) = \bigvee_{x \in X} T(f(x)) = \bigvee_{x \in X} (Tf)(x)$, therefore $T\exists f = \exists Tf$.

These results lead us the following definition:

1.4. DEFINITION. A monadic symmetric Boolean algebra (an algebra for short) is a system $(B, \exists, T)$, where $(B, \exists)$ is a monadic algebra, (B,T) is a symmetric algebra and $T\exists x = \exists Tx$, for $x \in B$.

As usual the unary operation $\forall x = -\exists -x$ defined on an algebra B, is called universal quantifier.

For an element k of an algebra B we say that k is a *constant* element if $\exists k = k$, and the set of such elements is denoted by $K(B)$. It is easy to see that $k \in K(B)$ if and only if $\forall x = x$ and that $\exists B = K(B)$.

If B is an algebra it is well-known that $K(B)$ is an upper conditionally complete Boolean subalgebra of B. If $x \in K(B)$, we have $\exists x = x$ and then

$\exists Tx = T\exists x = Tx$, therefore $Tx \in K(B)$. So $(K(B),T)$ is a subalgebra of the symmetric algebra (B,T).

Suppose (B,T) is a symmetric algebra and K is a symmetric subalgebra of B which is upper conditionally complete, then there exists a unique existential quantifier $\exists$ on B such that $(B,\exists,T)$ is an algebra and such that $\exists x = x$ for $x \in K$. If we define $\exists x = \bigwedge\{k \in K: x \leqslant k\}$, we know that $(B,\exists)$ is a monadic algebra such that $\exists x = x$ if and only if $x \in K$. All that remains is to verify that $T\exists x = \exists Tx$, $x \in B$.

a) $\exists Tx \leqslant T\exists x$. Endeed, from $x \leqslant \exists x$ it follows $Tx \leqslant T\exists x$ and hence $\exists Tx \leqslant \exists T\exists x$. Now $\exists x \in K$ and K is a symmetric subalgebra of B, then $T\exists x \in K$ and thus $\exists T\exists x = T\exists x$. It follows that $\exists Tx \leqslant T\exists x$.

b) $T\exists x \leqslant \exists Tx$. Since $\exists Tx = \bigwedge\{k \in K: Tx \leqslant k\}$, to prove that $T\exists x \leqslant \exists Tx$ we must show that $T\exists x \leqslant k$ for every $k \in K$ such that $Tx \leqslant k$. So let k be an element of K such that $Tx \leqslant k$. Then $x \leqslant Tk$ and hence $\exists x \leqslant \exists Tk$. It follows $T\exists x \leqslant T\exists Tk$. Since $k \in K$, then $Tk \in K$, thus $\exists Tk = Tk$ and hence $T\exists Tk = TTk = k$, therefore $T\exists x \leqslant k$.

For an element b of an algebra B we say that b is an *invariant* of B if $Tb = b$, and the set of such elements is denoted by $I(B)$. $(I(B),\exists)$ is a monadic subalgebra of the monadic algebra $(B,\exists)$. Finally $I(B) \cap K(B)$ is a Boolean subalgebra of B, where $Tx = \exists x = x$, $x \in I(B) \cap K(B)$. Moreover $I(B) \cap K(B) = I(K(B)) = K(I(B))$.

We proceed to consider some examples of algebras.

1.5. EXAMPLE. Let B be the Boolean algebra of Figure 1.

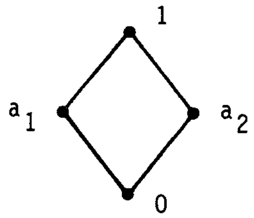


FIGURE 1

x	0	a_1	a_2	1
Tx	0	a_2	a_1	1
$\exists_1 x$	0	1	1	1
$\exists_2 x$	0	a_1	a_2	1

TABLE 1

Then $(B,\exists_1,T)$ is an algebra such that $K(B) = \{0,1\}$ and $I(B) = \{0,1\}$. $(B,\exists_2,T)$ is an algebra such that $K(B) = B$, $I(B) = \{0,1\}$.

1.6. EXAMPLE. Let B be the Boolean algebra of figure 2.

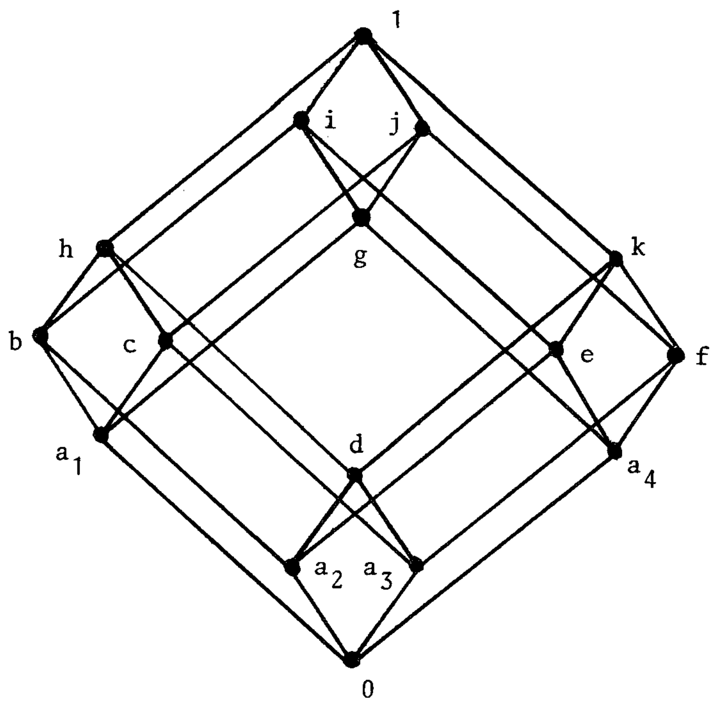


FIGURE 2

x	T_1x	T_2x	$\exists x$
0	0	0	0
a_1	a_1	a_2	b
a_2	a_2	a_1	b
a_3	a_4	a_4	a_3
a_4	a_3	a_3	a_4
b	b	b	b
c	g	e	h
d	e	g	h
e	d	c	i
f	f	f	f
g	c	d	i
h	i	i	h
i	h	h	i
j	j	k	1
k	k	j	1
1	1	1	1

TABLE 2

Then $(B,\exists,T_1)$ and $(B,\exists,T_2)$ are algebras.

2. HOMOMORPHISMS

2.1. Homomorphisms and quotient algebras

Let A,A' be algebras, $h:A \to A'$. We call h a homomorphism if:

H1) $h(x \wedge y) = h(x) \wedge h(y)$; H2) $h(-x) = -h(x)$

H3) $h(\exists x) = \exists h(x)$; H4) $h(Tx) = Th(x)$.

From H1 and H2 it follows that h is a Boolean homomorphism (B-homomorphism), from H1, H2 and H3, h is a monadic homomorphism (M-homomorphism) and from H1,H2 and H4, h is a symmetric homomorphism (S-homomorphism).

From H1 to H4 it is possible to prove:

H5) $h(x \vee y) = h(x) \vee h(y)$; H6) $h(0) = 0$

H7) $h(1) = 1$; H8) $h(\forall x) = \forall h(x)$.

The notions of epimorphism, monomorphism, isomorphism and homomorphic images are defined in the usual way.

Let A, A' be algebras and h a homomorphism from A into A'; the set $Ker(h) = \{x \in A: h(x) = 1\}$ has the following properties:

N1) $Ker(h)$ is a filter,
N2) If $x \in Ker(h)$ then $\forall x \in Ker(h)$,
N3) If $x \in Ker(h)$ then $Tx \in Ker(h)$.

If F is a filter of an algebra A and F verifies conditions N2 and N3 we say that F is a *monadic* S-*filter* (MS-*filter*, for short). If F is a filter verifying N2 we say that F is an M-filter. If a filter verifies N3 is said to be an S-filter.

2.1.1. LEMMA. Let A,A' be algebras and h a homomorphism from A onto A'. Then:

1 - The restriction $h_{|K(A)}$ of h to K(A) is an S-homomorphism from K(A) onto K(A'), such that $Ker(h_{|K(A)}) = K(A) \cap Ker(h)$.

2 - The restriction $h_{|I(A)}$ of h to I(A) is an M-homomorphism from I(A) onto I(A') such that $Ker(h_{|I(A)}) = I(A) \cap Ker(h)$.

If F is an MS-filter of an algebra B, then the relation "$x \equiv y$ (mod.F) if and only if there exists $f \in F$ such that $x \wedge f = y \wedge f$" is a congruence. If $x \in B$, $|x|$ denotes the congruence class containing x and B/F denotes the quotient algebra, where the operations are defined as usual:
$|x| \wedge |y| = |x \wedge y|$; $|x| \vee |y| = |x \vee y|$; $-|x| = |-x|$; $T|x| = |Tx|$; $\exists|x| = |\exists x|$. The function $\varphi: B \to B/F$ defined by $\varphi(x) = |x|$ is an epimorphism such that $Ker(\varphi) = F$.

2.1.2. LEMMA. Let B,B',B" be algebras, $h': B \to B'$ an epimorphism, $h'': B \to B''$ a homomorphism. If $Ker(h') \subseteq Ker(h'')$ then there exists a unique homomorphism $h: B' \to B''$ such that $h'' = h \circ h'$. Moreover if h" is an epimorphism, then h is an epimorphism. If h" is an epimorphism and $Ker(h') = Ker(h'')$ then h is an isomorphism.

2.1.3. COROLLARY. If B,B' are algebras and $h: B \to B'$ is an epimorphism, then B' and B/Ker(h) are isomorphic.

Hence, every homomorphic image of an algebra B can be obtained up to isomorphism, as a quotient B/F, where F is an MT-filter of B.

Recall that if X is a non-empty subset of a distributive lattice R with 0 and 1, then the filter $F(X)$ generated by X is the set of all elements $y \in R$ such that there exist elements $x_1, x_2, \ldots, x_n \in X$ such that $x_1 \wedge x_2 \wedge \ldots \wedge x_n \leqslant y$. It is well known that if X verifies the property "$x, y \in X$ implies $x \wedge y \in X$" then

$$F(X) = \{y \in R: \text{there exists } t \in X \text{ with } t \leqslant y\}.$$

If $X = \emptyset$, then $F(\emptyset) = \{1\}$. If $X = \{a\}$ we write $F(a)$ instead of $F(\{a\})$. $F(a)$ is called a principal filter. If R is finite, every filter is principal.

If B is a monadic algebra, the M-filter $MF(X)$ of B generated by X verifies $MF(X) = F(\forall X)$. If B is a symmetric algebra, the S-filter $SF(X)$ of B generated by X verifies $SF(X) = F(X \cup TX)$. Again we write $MF(a)$ instead of $MF(\{a\})$ and $SF(a)$ instead of $SF(\{a\})$. We have $MF(a) = F(\forall a)$ and $SF(a) = F(\{a\} \cup \{Ta\}) = F(\{a, Ta\}) = F(a \wedge Ta)$. It is easy to see that $F(a)$ is an M-filter if and only if $a \in K(B)$ and $F(a)$ is an S-filter if and only if $a \in I(B)$.

If B is an algebra, $MSF(X)$ denotes the MS-filter generated by X. We are going to prove that $MSF(X) = F(\forall X \cup T\forall X)$. From the preceding results $F(\forall X \cup T\forall X) = SF(\forall X)$. If $X = \emptyset$, then $MSF(\emptyset) = \{1\}$ and $SF(\forall\emptyset) = SF(\emptyset) = \{1\}$. Suppose now $X \neq \emptyset$. We know that $X \subseteq MSF(X)$, then $\forall X \subseteq \forall(MSF(X)) \subseteq MSF(X)$. Since $MSF(X)$ is an S-filter, we therefore have $TS(\forall X) \subseteq MSF(X)$. To prove that $MSF(X) \subseteq SF(\forall X)$ it is sufficient to show that 1) $X \subseteq SF(\forall X)$ and 2) $SF(\forall X)$ is an MS-filter.

1) If $y \in X$, $\forall y \in \forall X \subseteq SF(\forall X)$. Since $\forall y \leqslant y$ and $SF(\forall X)$ is a filter then $y \in SF(\forall X)$.

2) Since $SF(\forall X)$ is an S-filter, it remains to prove that it is also an M-filter. If $z \in SF(\forall X) = F(\forall X \cup T\forall X)$, then

$$a = y_1 \wedge y_2 \wedge \ldots \wedge y_n \leqslant z \text{ , where } y_i \in \forall X \cup T\forall X \text{ , } \quad i = 1, 2, \ldots, n \text{ ,}$$

then $y_i = \forall x$ with $x \in X$, or $y_i = T\forall x$ with $x \in X$. Then in any case $y_i \in K(B)$. Thus $a = \forall a \leqslant \forall z$ and consequently $\forall z \in F(\forall X \cup T\forall X) = SF(\forall X)$, which completes the proof.

In the case $X = \{a\}$, then $MSF(a) = MSF(\{a\}) = F(\{\forall a, T\forall a\}) = F(\forall a \wedge T\forall a)$.

We now give the relationship between MS-filters in an algebra B, S-filters in $K = K(B)$, M-filters in $I = I(B)$ and filters in $I(K(B))$. Let $\mathbf{D}, \mathbf{S}, \mathbf{M}$ and $\mathbf{F}$ respectively denote the set of all MS-filters in an algebra B, the set of all S-filters in $K(B)$, the set of all M-filters in $I(B)$ and the set of all filters in $I(K(B))$.

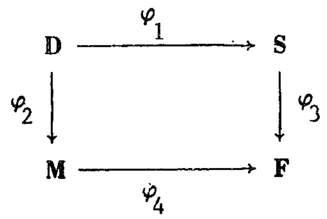
Consider the following functions:

$\varphi_1 : \mathbf{D} \to \mathbf{S}$, $\varphi_1(D) = D \cap K(B) = \exists D$

$\varphi_2 : \mathbf{D} \to \mathbf{M}$, $\varphi_2(D) = D \cap I(B)$

$\varphi_3 : \mathbf{S} \to \mathbf{F}$, $\varphi_3(F) = F \cap K(I(B)) = \exists F$

$\varphi_4 : \mathbf{M} \to \mathbf{F}$, $\varphi_4(F) = F \cap I(K(B))$



(1)

We have:

2.1.4. LEMMA. If we order the sets $\mathbf{D}, \mathbf{S}, \mathbf{M}$ and $\mathbf{F}$ by inclusion then φ_1, φ_2, φ_3 and φ_4 are order isomorphisms and the diagram (1) commutes.

PROOF. It is well known that φ_3 and φ_4 are isomorphisms. If $D \in \mathbf{D}$, then it is easy to see that $\varphi_1(D) = D \cap K(B) \in \mathbf{S}$, $D \cap K(B) = \exists D$ and $F(D \cap K(B)) = D$.

Conservely, if $D^* \in \mathbf{S}$, then $D = F(D^*) \in \mathbf{D}$ and $D^* = D \cap K(B)$. Since $\forall D^* = D^*$ and $TD^* \subseteq D^*$ we have $F(D^*) = MSF(D^*)$.

If $D \in \mathbf{D}$, then $\varphi_2(D) = D \cap I(B) \in \mathbf{M}$ and $D = F(D \cap I(B))$. Also, if $D' \in \mathbf{M}$ then $D = F(D') \in \mathbf{D}$ and $D' = D \cap I(B)$. Moreover $F(D') = MSF(D')$. The verification of commutativity being immediate, the proof is complete. ∎

2.1.5. LEMMA. If $D \in \mathbf{D}$ then

$$K(B/D) \cong K(B)/(D \cap K(B)),$$
$$I(B/D) \cong I(B)/(D \cap I(B)),$$
$$K(B/D) \cap I(B/D) \cong (I(B) \cap K(B))/(D \cap I(B) \cap K(B)).$$

In the next section we shall use these results.

2.2. Simple algebras and representation theorem

An algebra is called trivial if it has only one element.

2.2.1. DEFINITION. An algebra B is called simple if

1) B is non trivial.
2) All the homomorphic images of B are either trivial or isomorphic to B.

Since the homomorphic images of B are the algebras B/F, where F is an MS-filter, we have:

2.2.2. LEMMA. An algebra B is simple if and only if its only MS-filters are $\{1\}$ and B.

The proofs of the following lemmas is routine:

2.2.3. LEMMA. F(x) is an MS-filter of an algebra B if and only if $x \in I(K(B))$.

2.2.4. LEMMA. If M is an MS-filter of an algebra B, then B/M is simple if and only if M is maximal.

2.2.5. LEMMA. F(a) is a maximal MS-filter of B if and only if a is an atom of the Boolean algebra I(K(B)).

2.2.6. COROLLARY. a is an atom of the Boolean algebra I(K(B)) if and only if B/F(a) is a simple algebra.

2.2.7. THEOREM. If B is a non trivial algebra, then the following conditions are equivalent.

(i) B is a simple algebra.
(ii) I(K(B)) is a simple Boolean algebra.
(iii) K(B) is a simple symmetric algebra.
(iv) I(B) is a simple monadic algebra.

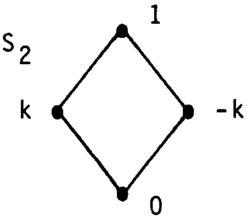
PROOF. (i) implies (ii) by lemmas 2.2.3 and 2.2.2. The equivalence of (ii) and (iii) was proved in [2, Th.2.2, page 209]. If $x \in I(B)$ then $\exists x \in K(I(B)) = I(K(B))$. It is then clear that (ii) implies (iv). Let (iv) hold, and let F be an MS-filter in B such that $F \neq \{1\}$. Then there exists $x \in F$, $x \neq 1$. The element $y = \forall x \wedge T\forall x \in F$ and also $y \in I(B)$. If $y = 1$, from $y \leqslant \forall x \leqslant x$ it would follow $x = 1$. Then $y \neq 1$ and therefore $\forall y = 0$. But it is clear that $\forall y = y$, thus we have $0 = y \in F$ and then $F = F(0)$. So (i) holds.
∎

It is known [2,10,11] that any simple symmetric algebra is isomorphic to one of the algebras listed in figure 3.

S_1

x	Tx
0	0
1	1

S_2



x	Tx
0	0
k	-k
-k	k
1	1

FIGURE 3

So an algebra B is simple if and only if either $K(B) \cong S_1$ or $K(B) \cong S_2$.

A simple algebra B is said to be of type I, if $K(B) = \{0,1\}$, and of type II, if $K(B) = \{0,k,-k,1\}$, where $k \neq 0,1$ and $Tk = -k$.

Certain intervals in an algebra B turn out to be algebras.

2.2.8. LEMMA. i) Let p,u belong to $I(K(B))$, such that $p \leqslant u$, then $([p,u],\vee,\wedge,\boldsymbol{-},p,u,\exists,T)$, is an algebra, where $\boldsymbol{-}x = p \vee (-x \wedge u)$, for $p \leqslant x \leqslant u$.

ii) $K([p,u]) = K(B) \cap [p,u]$.

iii) Let $a \in I(K(B))$. Then the function H defined by $H(x) = x \wedge a$ is a homomorphism of B onto $[0,a]$, such that $Ker(H) = F(a)$.

iv) If B is finite, then $A([0,a]) = A(B) \cap [0,a]$.

2.2.9. COROLLARY. $B/F(a)$ and $[0,a]$ are isomorphic algebras and then $K(B/F(a))$ and $K([0,a])$ are isomorphic symmetric algebras.

Let B be an algebra, we denote by $A(K \cap I)$ the set of all atoms of the Boolean algebra $K(B) \cap I(B)$ and by $A(K)$ the set of all atoms of the symmetric algebra $K(B)$.

2.2.10. LEMMA. 1) $[0,a]$ is simple of type I if and only if $a \in A(K \cap I) \cap A(K)$.

2) $[0,a]$ is simple of type II if and only if $a \in A(K \cap I) - A(K)$.

PROOF. 1) If $[0,a]$ is a simple algebra of type I then $K([0,a]) = \{0,a\}$. If $b \in K(B)$ and $0 \leqslant b \leqslant a$, then $b \in K(B) \cap [0,a] = K([0,a]) = \{0,a\}$. Hence $b = 0$ or $b = a$, and then $a \in A(K)$. From 2.2.6 and 2.2.9 it follows that $a \in A(K \cap I)$.

For the conserve, if $a \in A(K \cap I) \cap A(K)$ then 2.2.6 and 2.2.9 imply that $[0,a]$ is a simple algebra. If $b \in K([0,a]) = K(B) \cap [0,a]$ then from $a \in A(K)$ we have $b = 0$ or $b = a$. Therefore $K([0,a]) = \{0,a\}$, from which we conclude that $[0,a]$ is a simple algebra of type I.

2) If $[0,a]$ is a simple algebra of type II, from 2.2.6 and 2.2.9 we have $a \in A(K \cap I)$ and $K([0,a]) = K(B) \cap [0,a] = \{0,k,-k = -k \wedge a,a\}$, so it is clear that $a \notin A(K)$.

Conservely if $a \in A(K \cap I) - A(K)$ then from 2.2.6 and 2.2.9 we have that $[0,a]$ is a simple algebra. If $[0,a]$ were of type I, then from 1) it would follow that $a \in A(K)$, a contradiction. Therefore $[0,a]$ is of type II. ∎

Observe that if $b \in A(K)$ then $Tb \in A(K)$.

It is easy to see that:

2.2.11. LEMMA. $A(K \cap I) - A(K) = \{a \in A(K \cap I): a = b \vee Tb,\ b \in A(K),\ b \neq Tb\}$.

Observe that in this case since $b \notin I = I(B)$ then $b \notin A(K \cap I)$. Similarly $Tb \notin A(K \cap I)$. Then $b, Tb \in A(K) - A(K \cap I)$.

2.2.12. NOTE. Let B be a finite algebra. Then:

1) If $a \in A(K \cap I) - A(K)$ then a is the supremum of $2r$ atoms of B. Endeed $A(a) = A(b) \cup A(Tb)$, $A(b) \cap A(Tb) = \emptyset$. Hence $|A(a)| = |A(b)| + |A(Tb)|$. Besides T is a bijective correspondence between $A(b)$ and $A(Tb)$, then $|A(b)| = |A(Tb)|$. Therefore $|A(a)| = 2|A(b)| = 2r$, r natural, $r \geqslant 1$. From this we have that if B is a simple finite algebra of type II, then B has an even number of atoms. Endeed from 2.2.7 $I(K(B)) = \{0,1\}$, thus $A(K \cap I) = \{1\}$ and from $K(B) = \{0,k,-k,1\}$ it follows $A(K) = \{k,-k\}$. Then $A(K \cap I) - A(K) = \{1\}$ and so $|A(B)| = |A(1)| = 2r$.

2) If the cardinals $|A(K)|$ and $|A_1|$ are known, where $A_1 = A(K \cap I) \cap A(K)$, then from 2.2.11 and $A_1 \subseteq A(K)$ we have

$$|A(K \cap I) - A(K)| = \frac{|A(K)-A_1|}{2} = \frac{1}{2}(|A(K)| - |A_1|).$$

∎

2.2.13. THEOREM. If B is an algebra, the intersection of all maximal MS-filters in B is the MS-filter $\{1\}$.

PROOF. From lemma 2.1.4 the function $\varphi = \varphi_3 \circ \varphi_1$ is an isomorphism between the ordered sets **D** and **F**. Since in a Boolean algebra the intersection of all maximal filters is the filter {1} it follows that the intersection of all maximal MS-filters in the MS-filter {1}. ■

It follows from this theorem and using well-known general results on universal algebra [5, Corollary 1, Theorem 11, Chapter VI] that any algebra B with more than one element is a subdirect product of the family $\{B/M\}_{M\in M(B)}$, where $M(B)$ is the set of all maximal MS-filters in B. Moreover M being maximal, the algebras B/M are simple. In particular, any non trivial algebra B is semisimple.

As a consequence of more general results shown by A.Figallo [6], we can state that a finite symmetric algebra is uniquely determined, up to isomorphisms, by the number of its atoms and of its non-invariant atoms.

In what follows we identify isomorphic algebras and isomorphic symmetric algebras.

If the algebra B is finite, then B is a direct product of simple algebras, more precisely we have:

2.2.14. THEOREM. If B is a non trivial finite algebra then $B = \prod_{i=1}^{n} B/F(a_i)$, where $\{a_i\}_{1\leq i\leq n}$ is the family of all atoms of $I(K(B))$.

PROOF. We know that B is isomorphic to a subalgebra of the direct product $\prod_{i=1}^{n} B/F(a_i)$. This isomorphism is $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$ where $h_i\colon B \to B/F(a_i)$ are the natural homomorphism, $1 \leqslant i \leqslant n$. Let us prove that h is onto.

If $y = (y_1, y_2, \ldots, y_n) \in \prod_{i=1}^{n} B/F(a_i)$, for each $y_i \in B/F(a_i)$ there exists $x_i \in B$ such that $h_i(x_i) = y_i$. Consider the element $x = \bigvee_{i=1}^{n} (x_i \wedge a_i) \in B$. It is clear that $h_j(a_i) \in I(K(B/F(a_j)) = \{0,1\}$. If for $j \neq i$ $h_j(a_i) = 1$ then $a_i \in h_j^{-1}(\{1\}) = F(a_j)$ and then $a_j \leqslant a_i$, a contradiction. Therefore $h_j(a_i) = 0$ for $j \neq i$ and $h_j(a_j) = 1$. Then $h_j(x) = \bigvee_{i=1}^{n} (h_j(x_i) \wedge h_j(a_i)) =$ $= h_j(x_j) \wedge h_j(a_j) = h_j(x_j) \wedge 1 = h_j(x_j) = y_j$. Therefore $h(x) = y$. ■

If we write $K = K(B)$, $I = I(B)$, $A_1 = A(K \cap I) \cap A(K)$, $A_2 = A(K \cap I) - A(K)$

then $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A(K \cap I)$ and $B = \prod_{a \in A(K \cap I)} B/F(a) =$

$= \prod_{a \in A_1} B/F(a) \times \prod_{a \in A_2} B/F(a)$, where the quotients $B/F(a)$, $a \in A_1$ are simple algebras of type I and the quotients $B/F(a)$, $a \in A_2$ are simple algebras of type II.

2.3. Number of automorphisms in the finite simple algebras.

We obtained in the above section that finite simple algebras of type II have 2t atoms. This result is a consequence of more general results which we prove in this section. Also we determine the number of automorphisms in the finite simple algebras. This will be used en Chapter 4.

2.3.1. LEMMA. If B,C are finite Boolean algebras and h is Boolean homomorphism from B onto C, then:

(I) For $b \in A(C)$ there exists a unique $a \in A(B)$ such that $h(a) = b$ [3, lemma 2].

(II) If $a \in A(B)$ then either $h(a) \in A(C)$ or $h(a) = 0$.

(III) If in addition h is injective, then $h(a) \in A(C)$ for every $a \in A(B)$.

PROOF. (II) Suppose $h(a) \neq 0$. Then there exist $b_1, b_2, \ldots, b_t \in A(C)$ such that $h(a) = \bigvee_{i=1}^{t} b_i$.

From (I) we know that for each b_i there exists a unique $a_i \in A(B)$ such that $h(a_i) = b_i$ then $h(a) = \bigvee_{i=1}^{t} b_i = \bigvee_{i=1}^{t} h(a_i)$. Hence $h(a \wedge a_i) =$
$= h(a) \wedge h(a_i) = (\bigvee_{i=1}^{t} h(a_i)) \wedge h(a_i) = h(a_i) = b_i$, for each i, $1 \leqslant i \leqslant t$.
Since $b_i \neq 0$ then $a \wedge a_i \neq 0$ and therefore $0 < a \wedge a_i \leqslant a$. Then $a \wedge a_i = a$ and $h(a) = h(a \wedge a_i) = b_i \in A(C)$.

(III) It is an inmediate consequence of (II) and the hypothesis. ■

We denote by B_m the Boolean algebra with m atoms, $m \geqslant 1$.

2.3.2. LEMMA. Let $i,j \in B_m$ and let h be a Boolean automorphism of B_m such that $h(i) = j$. Then $h(A(i)) = A(j)$ and $h(A(-i)) = A(-j)$.

PROOF. If $i=0$, then $j=0$, and $h(A(0)) = \emptyset = A(0)$ and $h(A(-0)) = h(A(1)) = A(1) = A(-0)$.

Suppose that $a \in h(A(i))$, that is $a = h(b)$ with $b \in A(i)$. Since $b \in A(B_m)$ then from lemma 2.3.1 (III) we have $a \in A(B_m)$. From $b \leqslant i$ it follows that $a = h(b) \leqslant h(i) = j$, that is $a \in A(j)$, then $h(A(i)) \subseteq A(j)$.

Let $c \in A(j)$. From lemma 2.3.1 (I) there exists a unique $a \in A(B_m)$ such that $h(a) = c$. Since $a \leqslant 1 = i \vee -i$ and a is a prime element of B_m we have $a \leqslant i$ or $a \leqslant -i$. If $a \leqslant -i$ then $c = h(a) \leqslant h(-i) = -h(i) = -j$. Besides $c \leqslant j$, then $c \leqslant j \wedge -j = 0$ which is a contradiction. Then $a \leqslant i$ and therefore $c = h(a) \in h(A(i))$.

The other identity is obvious in view of the preceding proof and $h(-i) = -j$. ∎

It is well known [14,15,3] that if h is a Boolean automorphism of a finite Boolean algebra B_m, $m \geqslant 1$, then the function $f_h\colon A(B_m) \to A(B_m)$ defined by $f_h(a) = b$ if and only if $h(b) = a$, is a bijection.

Observe that f_h is the inverse of the restriction of h to $A(B_m)$. Converselly, if f is a bijection of $A(B_m)$ then the function $H_f\colon B_m \to B_m$ defined by:

$$H_f(x) = \vee\{a \in A(B_m)\colon f(a) \leqslant x\}$$

is a Boolean automorphism of B_m.

Let the set of all bijections of $A(B_m)$ be denoted by $F = BF(A(B_m))$ and let $B\text{-}AUT(B_m)$ stand for the set of all Boolean automorphisms of B_m. Then the function $\psi\colon F \to B\text{-}Aut(B_m)$ defined by $\psi(f) = H_f$, $f \in F$ is bijective. Then

$$|B\text{-}AUT(B_m)| = m! = 1 \times 2 \times \ldots \times m.$$

From Lemma 2.3.2 we conclude:

2.3.3. LEMMA. If $i \in B_m$ and $h \in B\text{-}AUT(B_m)$ is such that $h(i) = j$ then $f_h(A(j)) = A(i)$ and $f_h(A(-j)) = A(-i)$.

Observe that every bijection of $A(B_m)$ verifying one of these conditions, also verifies the other.

Let $F^* = BF^{(j,i)}(A(B_m))$ denote the set of all bijections f of $A(B_m)$ such that $f(A(j)) = A(i)$. The set of all Boolean automorphisms of B_m such that

$h(i) = j$ is denoted by $B\text{-}AUT^{(i,j)}(B_m)$. It is clear that $F^* \subseteq F$ and $F^* \neq \emptyset$ if and only if $|A(j)| = |A(i)|$.

The restriction $\psi^* = \psi|_{F^*}$ of ψ to the set F^* is a bijection between F^* and $B\text{-}AUT^{(i,j)}(B_m)$. Endeed, if $f \in F^*$ then $H_f(i) = \vee\{a \in A(B_m): f(a) \leqslant i\}$. If $a \in A(B_m)$ and $f(a) \leqslant i$ then $f(a) \in A(i) = f(A(j))$. Then $f(a) = f(c)$ with $c \in A(j)$. Since f is one-to-one then $a=c$ and then $a \in A(j)$. So $a \in A(B_m)$ and $a \leqslant j$. On the other hand, if $a \in A(B_m)$ and $a \leqslant j$ then $a \in A(j)$ and then $f(a) \in f(A(j)) = A(i)$. Thus $f(a) \leqslant i$. Therefore $H_f(i) = \vee\{a \in A(B_m): a \leqslant j\} = j$. Hence $H_f \in B\text{-}AUT^{(i,j)}(B_m)$.

Conservely, if $h \in B\text{-}AUT^{(i,j)}(B_m)$ we know that the function $f_h \in F$, defined by $f_h(a) = b$ if and only if $h(b) = a$, $a,b \in A(B_m)$, verifies $\psi(f_h) = h$ and in view of the remarks made earlier we can state that $f_h \in F^*$, then $\psi^*(f_h) = h$.

It is clear that ψ^* is one-to-one and then we have that ψ^* is a bijection between F^* and $B\text{-}AUT^{(i,j)}(B_m)$.

Then $|B\text{-}AUT^{(i,j)}(B_m)| = |F^*|$.

If $r = |A(j)| = |A(i)|$, it is clear that

$$|F^*| = |A(j)|! \, . \, |A(-j)|! = r! \, . \, (m-r)!$$

Note that if $i=j=0$ or $i=j=1$, then $B\text{-}AUT^{(0,0)}(B_m) = B\text{-}AUT(B_m) =$
$= B\text{-}AUT^{(1,1)}(B_m)$.

If $h \in B\text{-}AUT(B_m)$ is such that $h(i) = -i$, $i \in B_m$ then $|A(i)| = |A(-i)|$ and since $A(i) \cap A(-i) = \emptyset$ and $A(i) \cup A(-i) = A(B_m)$ then $|A(B_m)| =$
$= |A(i)| + |A(-i)| = 2|A(i)|$. Thus if B_m is a Boolean algebra such that there exists a Boolean automorphism h such that $h(i) = -i$, then m is even and $|A(i)| = \frac{m}{2}$. In this case:

$$|B\text{-}AUT^{(i,-i)}(B_m)| = (\tfrac{m}{2})! \, . \, (m - \tfrac{m}{2})! = ((\tfrac{m}{2})!)^2 .$$

If B is a simple finite algebra of type II, that is, $B = B_m$, m natural, $m \geqslant 2$, $K(B_m) = S_2 = \{0,k,-k,1\}$. Since T is a Boolean automorphism of B_m verifying $T(k) = -k$, we can state that B has an even number of atoms. In addition, from 2.3.2 it follows that $T(A(k)) = A(-k)$, then B has no invariant atoms.

Isomorphic algebras being identified, for each even natural number 2m, $m \geqslant 1$, there exists a unique simple algebra of type II, $B = B^{**}_{2m}$, where $A(B) = \{a_1, a_2, \ldots, a_{2m}\}$, $K(B) = \{0, k, -k, 1\}$, $k = \bigvee_{i=1}^{n} a_i$, $-k = \bigvee_{i=m+1}^{2m} a_i$, $Ta_i = a_{i+m}$, $1 \leqslant i \leqslant m$, $Ta_i = a_{i-m}$, $m+1 \leqslant i \leqslant 2m$.

2.3.4. LEMMA. If B is a simple algebra of type I then $h \in AUT(B)$ if and only if h is an S-automorphism of B.

PROOF. It is similar to the proof of lemma 2.4 indicated by L.Monteiro in [13]. ∎

If B is a finite simple algebra of type I with m atoms, and 2t non invariant atoms we note $B = B^{*}_{m,2t}$.

2.3.5. LEMMA. $|AUT(B^{*}_{r,2t})| = (r-2t)!.2^{t}.t!$.

PROOF. By lemma 2.3.4 $|AUT(B^{*}_{r,2t})| = |S\text{-}AUT(B^{*}_{r,2t})|$ and since $B^{*}_{r,2t}$ has (r-2t) invariant atoms and 2t non invariant atoms, from our results [3, Corollary 7] we have

$$|AUT(B^{*}_{r,2t})| = (r-2t)!.2^{t}.t!. \qquad ∎$$

2.3.6. LEMMA. If B is a simple algebra of type II, then $h \in AUT(B)$ if and only if h is an S-automorphism of B such that either $h(k) = k$ or $h(k) = -k$, where $K(B) = \{0, k, -k, 1\}$, $k \neq 0, 1$.

PROOF. If $h \in AUT(B)$, then h is an S-automorphism. Since $k \neq 0,1$ and h is an injective function then $h(k) \neq 0,1$. Besides $k \in K(B)$ implies $h(k) \in K(B)$ and then $h(k) \in \{k, -k\}$.

Conversely, let h be an S-automorphism of B such that $h(k) = -k$. It suffices to prove that

$$(*) \quad h(\exists x) = \exists h(x) \text{ for every } x \in B.$$

This condition is obviously verified if x=0. If x verifies $\exists x = k$ then $h(\exists x) = h(k) = -k$. Since $x \leqslant \exists x = k$ then $h(x) \leqslant h(k) = -k$. From $h(x) \not\leqslant 0$

and $h(x) \not\leq k$ it follows (see section 1) $\exists h(x) = \wedge\{y \in K(B): h(x) \leq y\} = = -k \wedge 1 = -k$. In a similar way if $\exists x = -k$ or $\exists x = 1$ we can prove that the condition (*) is verified. The case $h(k) = k$ being similar, the proof is complete. ∎

If $B = B^{**}_m$, m even, $m \geq 2$, is a finite simple algebra of type II, we denote by $S\text{-}AUT^{(k,k)}(B)$ the set of all S-automorphisms h of B such that $h(k) = k$. It is clear that this set is non empty. Similarly we denote by $S\text{-}AUT^{(k,-k)}(B)$ the set of all S-automorphisms h of B such that $h(k) = -k$. It is also clear that this set is non empty since $T \in S\text{-}AUT^{(k,-k)}(B)$.

From lemma 2.3.6 we can state that

$$AUT(B^{**}_m) = S\text{-}AUT^{(k,k)}(B) \cup S\text{-}AUT^{(k,-k)}(B)$$

and since $S\text{-}AUT^{(k,k)}(B) \cap S\text{-}AUT^{(k,-k)}(B) = \emptyset$ we have:

2.3.7. LEMMA. If B is a finite simple algebra of type II then $|AUT(B)| = |S\text{-}AUT^{(k,k)}(B)| + |S\text{-}AUT^{(k,-k)}(B)|$.

The following succession of results will be used in the calculus of the numbers of lemma 2.3.7.

If $h \in S\text{-}AUT^{(k,-k)}(B)$ and f_h is the function from A(B) into A(B) defined by $f_h(a) = b$ if and only if $h(a) = b$, we know that f_h is a bijection on A(B). From $h(k) = -k$ and lemma 2.3.3 we have that $f_h(A(-k)) = A(k)$, which is equivalent to $f_h(A(k)) = A(-k)$. Moreover, since h is an S-automorphism $f_h(Ta) = Tf_h(a)$, for every $a \in A(B)$, [3, lemma 3].

Let F** denote the set of all bijections on A(B) verifying 1) $f(A(k)) = = A(-k)$, and 2) $f(Ta) = Tf(a)$, for every $a \in A(B)$. If ψ^{**} is the restriction of ψ to the set F**, from 2) we have that H_f is an S-automorphism of B, and from 1) and the previous remarks $H_f(k) = -k$. Then it follows

$$|F^{**}| = |S\text{-}AUT^{(k,-k)}(B)|.$$

We are going to prove that the set BF(A(k),A(-k)) of all bijective functions from A(k) onto A(-k) and the set F** have the same cardinality.

If $f \in BF(A(k),A(-k))$ we define a bijection on A(B) in the following way:

$$F_f(a) = \begin{cases} f(a) & , \quad \text{if } a \in A(k) \\ Tf(a) & , \quad \text{if } a \in A(-k) \end{cases}$$

Since (i) $A(B) = A(k) \cup A(-k)$ and (ii) $\emptyset = A(k) \cap A(-k)$, F_f is a function defined on $A(B)$ and it is clear that it is bijective.

Since T is a Boolean automorphism on B verifying $Tk = -k$, then from lemma 2.3.2 we have $T(A(k)) = A(-k)$ and $T(A(-k)) = A(k)$. Observe that if $a \in A(k)$ then $F_f(a) \in A(-k)$ and if $a \in A(-k)$ then $Ta \in T(A(-k)) = A(k)$, thus $f(Ta) \in A(-k)$ and therefore $F_f(a) = Tf(Ta) \in T(A(-k)) = A(k)$.

F_f is onto. Endeed, if $b \in A(B)$, from (i) and (ii) we have (iii) $b \in A(k)$ or (iv) $b \in A(-k)$. In the case (iv) since f is onto there exists $a \in A(k)$ such that $f(a) = b$, then $T_f(a) = f(a) = b$. In the case (iii), from $b \in A(k)$ it follows that $a = Tb \in T(A(k)) = A(-k)$, then since f is onto there exists $c \in A(k)$ such that $f(c) = a$, then $d = Tc \in T(A(k)) = A(-k)$ and $F_f(d) = Tf(Td) = Tf(c) = Ta = b$. So we have that F_f is a bijection on $A(B)$. Moreover, $F_f(A(k)) = A(-k)$ and if $a \in A(k)$, since $Ta \in A(-k)$, then $F_f(Ta) = Tf(TTa) = Tf(a) = TF_f(a)$. If $a \in A(-k)$, then $Ta \in A(k)$ and $F_f(a) = Tf(Ta)$, thus $TF_f(a) = TTf(Ta) = f(Ta) = T_f(Ta)$. Then we have that $F_f \in F^{**}$. It is clear that if we define $\Phi(f) = F_f$ then Φ is a one-to-one function from $BF(A(k),A(-k))$ into F^{**}.

If $f \in F^{**}$, $f' = f_{|A(k)}$ is a bijection from $A(k)$ onto $A(-k)$. It is easy to see that $\Phi(f') = f$.

Then we have $|F^{**}| = |BF(A(k),A(-k)|$, and since $|A(k)| = \frac{m}{2}$, then $|S\text{-}AUT^{(k,-k)}(B)| = (\frac{m}{2})!$.

In a similar way we can prove that the set $S\text{-}AUT^{(k,k)}(B)$ and the set of all bijective functions on $A(k)$ have the same cardinality and therefore

$$|S\text{-}AUT^{(k,k)}(B)| = (\tfrac{m}{2})!.$$

Then if B^{**}_m is a finite simple algebra of type II, with m atoms, m even, $m \geqslant 2$

$$|AUT(B^{**}_m)| = 2(\tfrac{m}{2})!.$$

3. FINITELY GENERATED ALGEBRAS

Let G be a part of an algebra B; we denote by S(G) (respectively SS(G), MS(G)) the smallest subalgebra (symmetric subalgebra, monadic subalgebra) of B containing G. S(G) (SS(G),SM(G)) is called the subalgebra (symmetric subalgebra, monadic subalgebra) generated by G. It is clear that $SS(G) \subseteq S(G)$, $MS(G) \subseteq S(G)$.

The following lemma can be proved in the usual way:

3.1. LEMMA. If B,B' are algebras, h an homomorphism from B into B' and $G \subseteq B$, then $S(h(G)) = h(B)$.

3.2. LEMMA. If B is a simple algebra, and $G \subseteq B$ is such that $S(G) = B$ then:

(i) If B is of type I, $SS(G) = B$.

(ii) If B is of type II, $SS(G,k) = B$, where we write SS(G,k) instead of $SS(G \cup \{k\})$.

PROOF. (i) Since $K(B) = \{0,1\} \subseteq SS(G)$ then SS(G) is a subalgebra containing G. Thus $B = S(G) \subseteq SS(G)$, that is $B = SS(G)$.

(ii) We have $K(B) = \{0,k,-k,1\} \subseteq SS(G,k)$. So SS(G,k) is a subalgebra containing G. Hence $SS(G,k) = B$. ■

We now prove that if an algebra B has a finite set of generators, then B is finite. That is, if G is a finite subset of B with $|G| = n$ and $S(G) = B$, then B is finite.

We know that B is a subalgebra of the algebra $P = \prod_{M \in M(B)} S_M$, where $M(B)$ is the set of all maximal MS-filters of B and $S_M = B/M$, $M \in M(B)$. If we prove that B/M is finite for every $M \in M(B)$ and the set $M(B)$ is finite, then P is finite and hence B is finite.

3.3. LEMMA. Let B be an algebra, $G \subseteq B$ such that $|G| = n$, n natural, $n \geq 1$, $S(G) = B$ and $M \in M(B)$. Then $|B/M| \leq 2^{(4^{n+1})}$.

PROOF. We know from 2.2.4 that if $M \in M(B)$ then B/M is a simple algebra.

First case: B/M is simple of type I, that is $K(B/M) = \{0,1\}$. If h is the natural homomorphism from B onto B/M then by 3.1 and 3.2 $SS(h(G)) = S(h(G)) = B/M$. So h(G) is a set of generators of the symmetric algebra B/M. Since $|h(G)| \leq n$ we have that B/M is finite and $|B/M| \leq 2^{(4^n)}$ ([2], [11]).

Second case: B/M is a simple algebra of type II, that is $K(B/M) = \{0,k,-k,1\}$ where $k \neq 0,1$. Then $SS(h(G),k) = S(h(G)) = B/M$. So $H(G) \cup \{k\}$ is a set of generators of the symmetric algebra B/M.

Therefore $|B/M| \leqslant 2^{(4^{n+1})}$.

Then we have if B is a finitely generated algebra there exists only a finite number of algebras B/M, $M \in M(B)$. Endeed, following a similar path as in [12], let $w = |B/M|$, $M \in M(B)$, we have:

1) On a finite set it is possible to define only a finite number of structures of monadic symmetric Boolean algebra. Then for each $w \leqslant 2^{(4^{n+1})}$ there exists only a finite number of algebras B/M, with $|B/M| = w$.

2) From 3.3 it follows that for $w > 2^{(4^{n+1})}$ we have no algebras B/M with w elements.

Then, isomorphic algebras being identified, from 1) and 2) we have that there exists only a finite number of algebras B/M, $M \in M(B)$.

Let $S^1, S^2, \ldots, S^f$ be fixed algebras pairwise non isomorphic such that each algebra S^j is at least isomorphic to one of the algebras B/M, $M \in M(B)$ and such that every algebra B/M, $M \in M(B)$ is isomorphic to S^j, $1 \leqslant j \leqslant f$. If $M^j = \{M \in M(B) : B/M \cong S^j\}$ then $M(B) = \bigcup_{j=1}^{f} M^j$ and $M^i \cap M^j = \emptyset$ if $i \neq j$.

Then in order to prove that $M(B)$ is finite it is sufficient to prove that M^j is finite for every j.

Let $EPI(B,S^j)$ be the set of all homomorphisms from B onto S^j and let $F^*(G,S^j)$ stand for the set of all functions $f: G \to S^j$ such that $S(f(G)) = S^j$. We have that $F^*(G,S^j) \subseteq (S^j)^G$.

Let r be the function from $EPI(B,S^j)$ into $F^*(G,S^j)$ defined by $r(h) = h_{|G}$, for $h \in EPI(B,S^j)$. It is easy to see that r is one-to-one, and then

$$|EPI(B,S^j)| \leqslant |F^*(G,S^j)| \leqslant |(S^j)^G| < \infty.$$

Besides, the function $s: EPI(B,S^j) \to M^j$ defined by $s(h) = Ker(h)$ is onto. Endeed, if $M \in M^j$ let σ_M be the isomorphism between A/M and S^j and $h_M: B \to B/M$ the canonical homomorphism. Then the function $h = \sigma_M \circ h_M$ has kernel M, that is $s(h) = M$.

So $|M^j| \leqslant |EPI(B,S^j)|$ and then $M(B)$ is finite.

We have then proved that if B is finitely generated then B is finite. More precisely, if $AUT(S^j)$ is the set of all automorphisms of S^j then

$$|M^j| = \frac{|EPI(B,S^j)|}{|AUT(S^j)|} .$$

Endeed if $H \in EPI(B,S^j)$ and $h \in AUT(S^j)$ then $(h \circ H) \in EPI(B,S^j)$ and $s(h \circ H) = s(H)$. If $H_1,H_2 \in EPI(B,S^j)$ and $Ker(H_1) = Ker(H_2)$ then there exists $h \in AUT(S^j)$ such that $H_1 = h \circ H_2$, then $s^{-1}(M) = \{h \circ H: h \in AUT(S^j)\}$ where $M \in M(B)$ verifies $B/M = S^j$.

4. FREE ALGEBRAS

We start this section by recalling the following well-known definition:

4.1. DEFINITION. Given a cardinal number $c > 0$, we shall say that $\mathcal{L}$ is an algebra with c free generators if:

L1) There is a subset G of $\mathcal{L}$, of power c, such that $S(G) = \mathcal{L}$.

L2) Given an algebra B and an application f from G into B, there is a homomorphism $\overline{f}$, necessarily unique, from $\mathcal{L}$ into B such that $\overline{f}$ is an extension of f.

If it is so, we shall say that G is a set of free generators of $\mathcal{L}$. An algebra is said to be free if it has a set of free generators. We shall note $\mathcal{L} = L(c)$. Since the monadic symmetric algebras are defined by equations, we can state, by a G.Birkhoff theorem of universal algebra [5], the existence and uniqueness, up to isomorphisms, of $L(c)$. In view of the preceding paragraph, we can state that $L(n)$ is finite, for every natural number $n > 0$. Furthermore if $K(n) = K(L(n))$, $I(n) = I(L(n))$, $A_1 = A(K(n) \cap I(n)) \cap A(K(n))$; $A_2 = A(K(n) \cap I(n)) - A(K(n))$, then $L(n) = P_1(n) \times P_2(n)$ where $P_1(n) = \prod_{a\varepsilon A_1} L(n)/F(a)$; $P_2(n) = \prod_{a\varepsilon A_2} L(n)/F(a)$.

We know that if $a \in A_1$ then $L(n)/F(a)$ is a finite simple algebra of type I, $|L(n)/F(a)| \leqslant 2^{(4^n)}$ and then (see 2.3)

$$L(n)/F(a) = B^*_{r,2t} , \quad 1 \leqslant r \leqslant 4 \quad , \quad 0 \leqslant 2t \leqslant r.$$

If $a \in A_2$, $L(n)/F(a)$ is a finite simple algebra of type II, $|L(n)/F(a)| \leqslant 2^{(4^{n+1})}$ and $L/F(a) = B^{**}_{2t}$, $2 \leqslant 2t \leqslant 4^{n+1}$.

Hence

$$P_1(n) = \prod_{\substack{1\le r\le 4^n \\ 0\le 2t\le r}} [B^*_{r,2t}]^{\alpha^{(n)}_{r,2t}} \quad ; \quad P_2(n) = \prod_{2\le 2t\le 4^{n+1}} [B^{**}_{2t}]^{\alpha^{(n)}_{2t}}$$

where

$$\alpha^{(n)}_{r,2t} = \frac{|EPI(L(n),B^*_{r,2t})|}{|AUT(B^*_{r,2t})|} \quad \text{and} \quad \alpha^{(n)}_{2t} = \frac{|EPI(L(n),B^{**}_{2t})|}{|AUT(B^{**}_{2t})|}$$

A) Computation of $\alpha^{(n)}_{r,2t}$, $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant r$.

A1) Computation of $AUT(B^*_{r,2t})$, $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant r$.

By lemma 2.3.5 $|AUT(B^*_{r,2t})| = 2^t.(r-2t)!.t!$.

A2) Computation of $|EPI(L(n),B^*_{r,2t})|$, $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant r$.

We are going to prove that the symmetric subalgebra of $L(n)$ generated by G, noted $SS(G)$, is the symmetric algebra with n free generators $S(n)$.

Let C be a symmetric algebra and f an application from G into C. We define an existential quantifier on C by means of $\exists x = x$, $x \in C$. Then $T\exists x = \exists Tx$, and therefore $(C,\exists)$ is an algebra. Then f can be extended to a homomorphism h from $L(n)$ into C. It is clear that the restriction of h to $SS(G)$ is an S-homomorphism from $SS(G)$ into C extending f. So $SS(G) = S(n)$.

If $B^*_{r,2t}$, $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant r$, is a finite simple algebra of type I we denote by $S\text{-}EPI(S(n),B^*_{r,2t})$ the set of all S-epimorphisms from $S(n)$ onto $B^*_{r,2t}$.

4.2. LEMMA. $|EPI(L(n),B^*_{r,2t})| = |S\text{-}EPI(S(n),B^*_{r,2t})|$.

PROOF. Let B be a finite simple algebra of type I and $H \in EPI(L(n),B)$. We know that $S(n) = SS(G)$. If we consider the function $\Phi(H) = H_{|S(n)}$ it is clear that $h = H_{|S(n)}$ is an S-homomorphism from $S(n)$ into B. By lemma 3.1

$S(H(G)) = H(L(n)) = B$ and from lemma 3.2 $SS(H(G)) = B$. Since $S(n) = SS(G)$ then $B = SS(H(G)) = SS(h(G)) = h(S(n))$, therefore $h \in S\text{-}EPI(S(n),B)$.

If $H_1, H_2 \in EPI(L(n),B)$ are such that $\Phi(H_1) = \Phi(H_2)$ then $H_{1|G} = H_{2|G}$. Since $L(n)$ is a free algebra this function can be extended to a unique homomorphism from $L(n)$ into B, so $H_1 = H_2$.

If $h \in S\text{-}EPI(S(n),B)$, let H be the extension of the function $f = H_{|G}$ to $L(n)$. Then $H(L(n)) = S(H(G)) = SS(H(G)) = SS(f(G)) = SS(h(G)) = h(S(n)) = B$. Thus $H \in EPI(L(n),B)$ and $\Phi(H) = H_{|S(n)}$. Since h and $H_{|S(n)}$ coincide on G then $H_{|S(n)} = h$. ∎

4.3. COROLLARY. $B^*_{r,2t}$ is a homomorphic image of $L(n)$ if and only if $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant 4^n-2^n$ and $0 \leqslant r-2t \leqslant 2^n$.

PROOF. From 4.2, $B^*_{r,2t}$ is a homomorphic image of $L(n)$ if and only if the symmetric algebra $B^*_{r,2t}$ is a homomorphic image of the symmetric algebra $S(n)$. Since $S(n)$ has 4^n atoms, 4^n-2^n non invariant atoms and 2^n invariant atoms ([2]) then the symmetric algebra $B^*_{r,2t}$ ia a homomorphic image of the symmetric algebra $S(n)$ if and only if ([3]), $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant 4^n-2^n$ and $0 \leqslant r-2t \leqslant 2^n$. ∎

From our results in [3] we have

$$|S\text{-}EPI(S(n),B^*_{r,2t})| = 2^t \cdot V_{4^n-(4^n-2^n),\,r-2t} \cdot V_{\frac{4^n-2^n}{2},\,t}$$

where $1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant 4^n-2^n$, $0 \leqslant r-2t \leqslant 2^n$.

Therefore

$$\alpha^{(n)}_{r,2t} = \frac{2^t \cdot V_{2^n,\,r-2t} \cdot V_{\frac{4^n-2^n}{2},\,t}}{2^t \cdot (r-2t)! \cdot t!} = \binom{2^n}{r-2t} \cdot \binom{\frac{4^n-2^n}{2}}{t},$$

$1 \leqslant r \leqslant 4^n$, $0 \leqslant 2t \leqslant 4^n-2^n$, $0 \leqslant r-2t \leqslant 2^n$.

If $n = 1$, we have $1 \leqslant r \leqslant 4$, $0 \leqslant 2t \leqslant 2$, $0 \leqslant r-2t \leqslant 2$, and then $\alpha^{(1)}_{1,0} = 2$, $\alpha^{(1)}_{2,0} = 1$, $\alpha^{(1)}_{2,2} = 1$, $\alpha^{(1)}_{3,2} = 2$, $\alpha^{(1)}_{4,2} = 1$. Then

$P_1(1) = (B^{*}_{1,0})^2 \times B^{*}_{2,0} \times B^{*}_{2,2} \times (B^{*}_{3,2})^2 \times B^{*}_{4,2}$.

Hence $|P_1(1)| = 2^{16} = 65.536$. The Hasse diagrams of these factors as well as the operations T and $\exists$ are indicated in Figure 4, (page 24).

B) Computation of $\alpha^{(n)}_{2t}$, $2 \leqslant 2t \leqslant 4^{n+1}$.

The numbers $\alpha^{(n)}_{2t}$ will be determined in a different way from that used in the determination of $\alpha^{(n)}_{r,2t}$.

Let M(2n) denote the monadic algebra with a set

$$G = \{g_1, g_2, \ldots, g_n, g_{n+1}, g_{n+2}, \ldots, g_{2n}\}$$

of 2n free generators.

Let t: G $\rightarrow$ G be defined by

$$t(g_i) = \begin{cases} g_{n+i} & , \quad 1 \leqslant i \leqslant n \\ g_{i-n} & , \quad n+1 \leqslant i \leqslant 2n. \end{cases}$$

Then t can be extended to a unique M-homomorphism T from M(2n) into M(2n). From $G = t(G) = T(G) \subseteq T(M(2n)) \subseteq M(2n)$ and T(M(2n)) being an M-subalgebra of the monadic algebra M(2n) it follows that T(M(2n)) = M(2n), that is T is onto. Since M(2n) is finite we have that T is one-to-one.

It is clear that $M = \{x \in M(2n): TTx = x\}$ is an M-subalgebra of M(2n) containing G and then M = M(2n). Thus TTx = x for every $x \in M(2n)$.

Therefore we have that (M(2n),T) is an algebra.

We shall prove that $H = \{g_1, g_2, \ldots, g_n\}$ is a set of free generators of the algebra M(2n). This is a consequence of the results of Abad and Figallo [1], but we include here a demostration for the sake of completeness.

From $H \subseteq S(H)$ and S(H) being an S-subalgebra, for $h \in H$, $Th \in S(H)$, that is $TH \subseteq S(H)$. Then $G = H \cup TH \subseteq S(H)$ and since S(H) is an M-subalgebra of M(2n), then $M(2n) = SM(G) \subseteq S(H)$. Thus S(H) = M(2n).

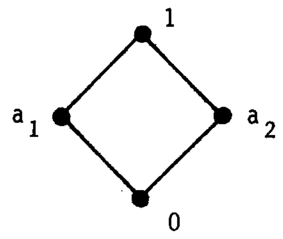
If B is an algebra and f is a function from H into B we define

$$F(g_i) = f(g_i) \quad \text{for} \quad i = 1,2,\ldots,n$$

$$F(g_{n+1}) = Tf(g_i) \quad \text{for} \quad i = 1,2,\ldots,n.$$

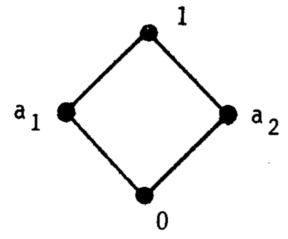
$B^*_{1,0}$

x	0	1
Tx	0	1
∃x	0	1



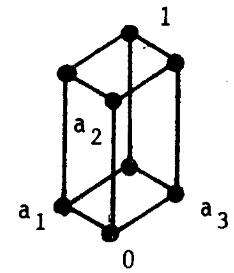
$B^*_{2,0}$

x	0	a_1	a_2
Tx	0	a_1	a_2
∃x	0	1	1



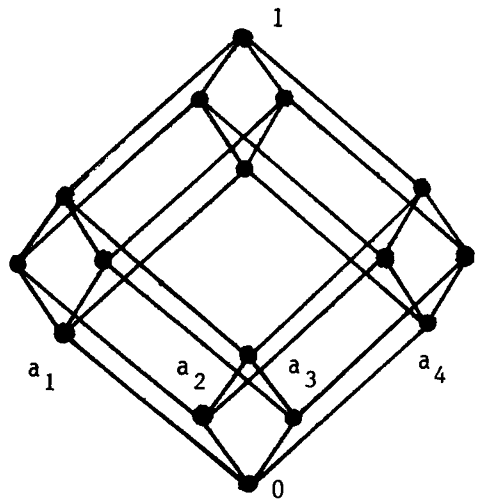
$B^*_{2,2}$

x	0	a_1	a_2
Tx	0	a_2	a_1
∃x	0	1	1



$B^*_{3,2}$

x	0	a_1	a_2	a_3
Tx	0	a_2	a_1	a_3
∃x	0	1	1	1



$B^*_{4,2}$

x	0	a_1	a_2	a_3	a_4
Tx	0	a_2	a_1	a_3	a_4
∃x	0	1	1	1	1

FIGURE 4

Then F is a function from G into B such that $F_{|H} = f$ and therefore F can be extended to an M-homomorphism $H_F: M(2n) \to B$.
It is easy to see that the set $S = \{x \in M(2n): H_F(Tx) = TH_F(x)\}$ is a subalgebra of the algebra M(2n) containing H, and then S = M(2n), that is $H_F(Tx) = TH_F(x)$ for every x in M(2n).
Then we have the following theorem:

4.4. THEOREM. (M(2n),T) is the algebra with n free generators.

That is, L(n) = M(2n) and therefore K(L(n)) = K(M(2n)). Then (see [13])

$$|L(n)| = 2^{[2^{2n}.2^{(2^{2n}-1)}]}.$$

Moreover K(Ln)) has $2^{(2^{2n})}-1 = 2^{4^n}-1$ atoms. In particular

$$|L(1)| = 2^{32} = 4.294.967.296.$$

We are going to determine the decomposition of L(n) as a direct product of simple algebras, which is a more precise result that 4.4.
To this end we only have to compute te numbers $\alpha_{2t}^{(n)}$, $2 \leqslant 2t \leqslant 4^{n+1}$.
This is equivalent to determine the elements $a \in A_2$ such that $|L(n)/F(a)| = 2^{2t}$, $2 \leqslant 2t \leqslant 4^{n+1}$. From L(n)/F(a) = [0,a], this is equivalent to determine the elements $a \in A_2$ which are supremum of 2t atoms, $2 \leqslant 2t \leqslant 4^{n+1}$, of the algebra L(n) = M(2n).
Since K(n) = K(L(n)) = K(M(2n)) we know from our results [4] that if $b \in A(K(n))$, then:
1) $1 \leqslant |A(b)| \leqslant 2^{2n} = 4^n$, and
2) there exist $\binom{4^n}{r}$ atoms of K(n) such that $|A(b)| = r$, $1 \leqslant r \leqslant 4^n$.
If $A_r(K(n)) = \{b \in A(K(n)): |A(b)| = r\}$, $1 \leqslant r \leqslant 4^n$, then A(K(n)) is the disjoint union of the sets $A_r(K(n))$, $1 \leqslant r \leqslant 4^n$. By 2.2.11 we know that $A_2 = \{a \in A(K(n)) \cap I(n)): a = b \vee Tb,\ b \in A(K(n)),\ b \neq Tb\}$, and the elements b,Tb belong to $A(K(n)) - A(K(n) \cap I(n))$.
Since for $b \in A(K(n))$ we have $1 \leqslant |A(b)| \leqslant 4^n$ then the elements of A_2 are supremum of at most 2.4^n atoms of the algebra L(n). Since $A_{4^n}(K(n))$ has

only one element, this element is an invariant, so the elements of A_2 are supremum of at most $2.(4^n-1)$ atoms of $L(n)$.

It is easy to see that the set $\{a \in A_2 : |A(a)| = 2r\}$ and $\{(b,Tb) : b \in A_r(K(n)) - A_1\}$, $2 \leqslant 2r \leqslant 2.(4^n-1)$, have the same cardinality. Therefore

$$\alpha_{2r}^{(n)} = \frac{1}{2}\,|A_r(K(n))-A_1| = \frac{1}{2}\,|A_r(K(n))-(A_r(K(n)) \cap A_1)| =$$

$$= \frac{1}{2}\,[\,|A_r(K(n))| - |A_r(K(n)) \cap A_1|\,] =$$

$$= \frac{1}{2}\,[\binom{4^n}{r} - \sum_{0\leq 2t\leq r} \alpha_{r,2t}^{(n)}] \quad , \quad 2 \leqslant 2r \leqslant 2.(4^n-1).$$

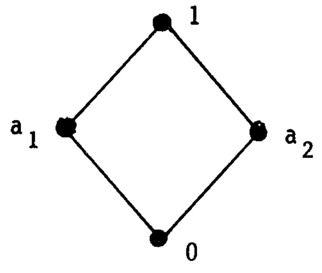
If $n=1$, we have $2 \leqslant 2r \leqslant 2.(4^1-1) = 6$, then

$\alpha_2^{(1)} = \frac{1}{2}[\binom{4}{1} - \alpha_{1,0}^{(1)}] = 1$; $\alpha_4^{(1)} = \frac{1}{2}[\binom{4}{2} - (\alpha_{2,0}^{(1)} + \alpha_{2,2}^{(1)})] = 2$;

$\alpha_6^{(1)} = \frac{1}{2}[\binom{4}{3} - \alpha_{3,2}^{(1)}] = 1$.

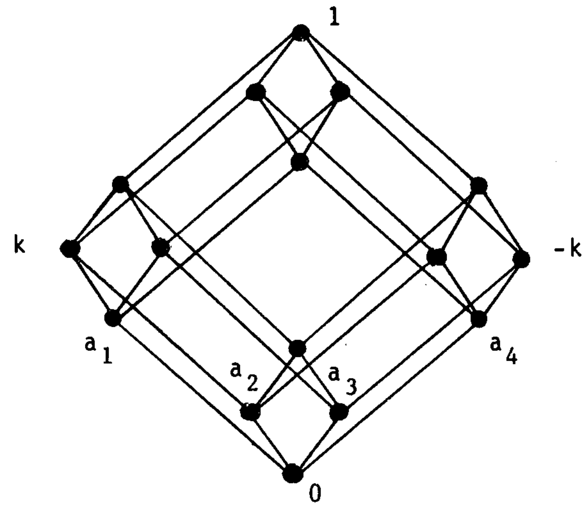
Then $P_2(1) = B_2^{**} \times (B_4^{**})^2 \times B_6^{**}$.

Hence $P_2(1) = 2^{16}$. The Hasse diagrams of these factors as well as the operations T and ∃ are indicated in figures 5 and 6 (pages 27 and 28). ■



B_2^{**}

x	0	a_1	a_2
Tx	0	a_2	a_1
∃x	0	a_1	a_2



B_4^{**}

x	0	a_1	a_2	a_3	a_4
Tx	0	a_3	a_4	a_1	a_2
∃x	0	k	k	-k	-k

FIGURE 5

x	0	a_1	a_2	a_3	a_4	a_5	a_6
Tx	0	a_4	a_5	a_6	a_1	a_2	a_3
∃x	0	k	k	k	-k	-k	-k

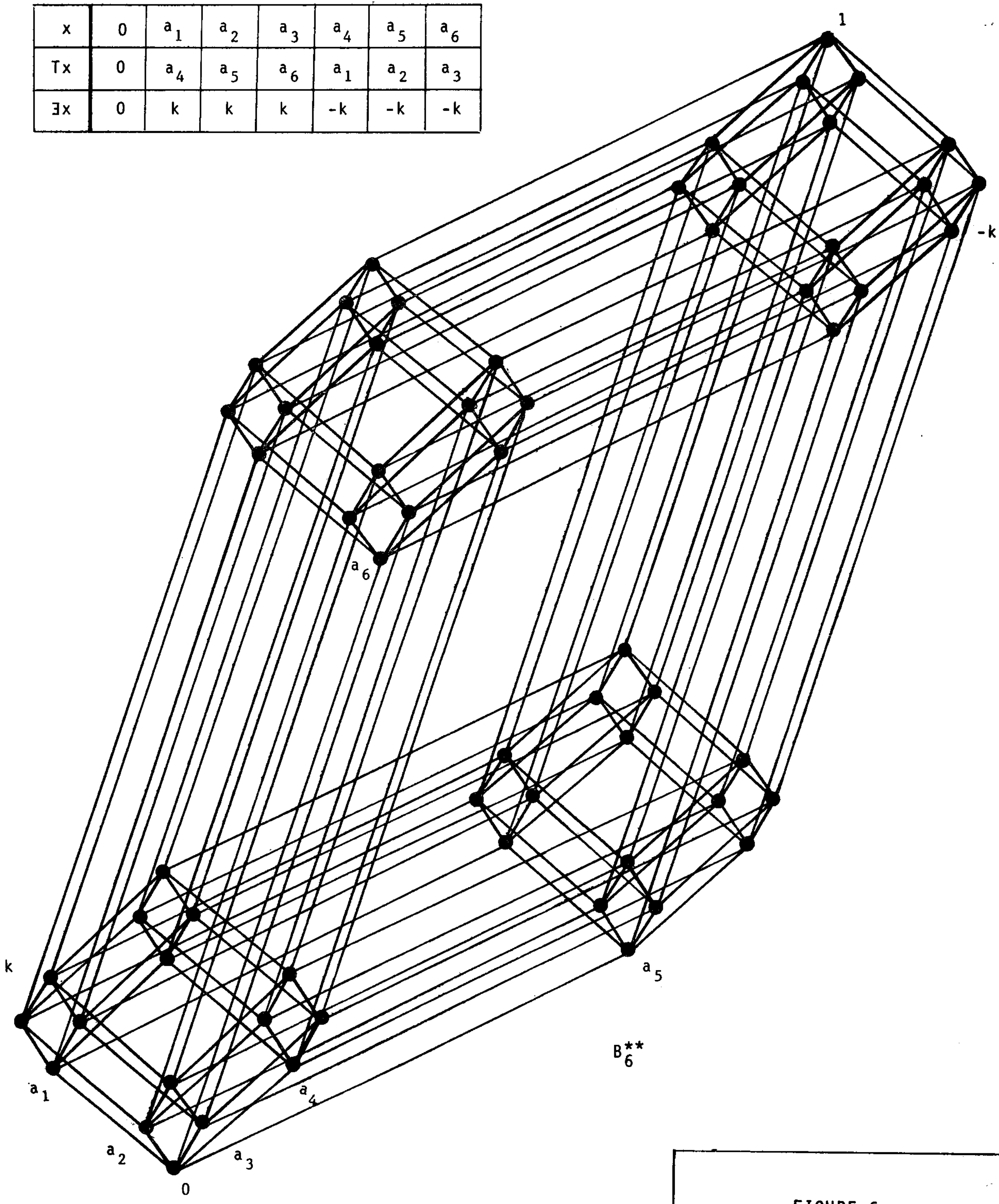


FIGURE 6

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