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Morgan lattices are defined as distributive lattices possessing a monary operation (−) which obeys the involutory law (−−a = a) and de Morgan laws. This structure has been studied by C. Moisil[3] (i), J. Kalman[2], and A. Monteiro[4]; an important particular case has also been considered by Byalinicki-Birula and Rasiowa[1] under the name of quasi-boolean algebras.

The purpose of this article is to characterise Morgan lattices by means of operations of infimum (Λ) and negation (−). (ii). This work is made easy by the use of Sholander's characterisation[5] of a distributive lattice as a non empty set with the binary operations (Λ)

(i) See the bibliographical references at the end of this article.
(ii) It is evident that a dual characterisation can be done using supremum (Λ) instead of infimum.
and (\lor), which fulfill the following axioms:

\[ S_1 \quad a = a \land (a \lor b) \]
\[ S_2 \quad a \land (b \lor c) = (c \land a) \lor (b \land a) \]

**THEOREM 1:** Let \( A \) be a non empty set, with the operations (\land) and (\lor). Let us define:

1) \( a \lor b = \neg (\neg a \land \neg b) \)

The system \((A, \land, \lor)\) is a Morgan lattice if and only if it obeys the following axioms:

\[ M_1 \quad a = a \land \neg (\neg a \land \neg b) \]
\[ M_2 \quad a \land \neg (b \land \neg c) = \neg (\neg (c \land a) \land \neg (b \land a)) \]

**DEMONSTRATION:** It is evident that \( M_1 \) and \( M_2 \) are necessary. To prove that they are sufficient, we shall first prove that \( A \) is a distributive lattice; i.e., that it obeys \( S_1 \) and \( S_2 \), which are readily verified.

\[ S_1 \quad a \land (a \lor b) = a \land \neg (\neg a \land \neg b) = a \]
\[ S_2 \quad (c \land a) \lor (b \land a) = \neg (\neg (c \land a) \land \neg (b \land a)) = \]
\[ \quad = a \land \neg (b \land \neg c) = a \land (b \lor c) \]

We must still prove:

2) \( \neg \neg a = a \)
3) \( \neg (a \lor b) = \neg a \land \neg b \)

As \( A \) is a distributive lattice, we have:

4) \( a \land a = a \)
5) \( a \land b = b \land a \)

From 5) and \( M_2 \) we obtain:
6) \[ a \land \neg(b \land \neg c) = \neg((b \land a) \land \neg(c \land a)) \]
and replacing \( b \) and \( c \) by \( a \) it results, by 4)

7) \[ a \land \neg(a \land \neg a) = \neg((a \land a) \land \neg(a \land a)) = \]
\[ = \neg(-a \land \neg a) = -a \]

and applying \( M_1 \) to the first member of 7), we get 2).

As to 3), it is immediately obtained from 1) and 2).

THEOREM 2: Postulates \( M_1 \) and \( M_2 \) are independent.

DEMONSTRATION: Let \( A \) be the set \( \{0, 1\} \). Let us define \( \neg a = a \), and let us define \( a \land b \) in two different ways:

1) \[ a \land b = a \]

ii) \[ a \land b = a \cdot b \]

In the first case, \((A, \land, \neg)\) verifies \( M_1 \), but not \( M_2 \). With the second definition, \( M_2 \) is satisfied, but not \( M_1 \).
BIBLIOGRAPHY


