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Bisimulations for Boolean Relevant Logics

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1 Introduction

The notion of Bisimulation is an important tool for the study of the model theory of classical modal logic and also plays a crucial role in Theoretical Computer Science (see [6],[4], [1] and [2]). This notion was introduced by J. van Benthem in [1] under the name of *p-relation.* Some very important results, as for example preservation and definibility results, are proved using bisimulation as the fundamental tool. In this direction the work [11] contains general results in the model theory of classical modal logic $\mathcal{ML}$ where the role of the bisimulation is fundamental.

Routley and Meyer [7] provide a Kripke-style semantics for some Boolean relevant logics using a ternary accessibility relation. This semantic is an adaptation of the relational semantics for Relevant logics (see [10]). With a ternary relation defined on a set X we can define two binary operations on the power set algebra $\mathcal{P}(X)$: an operator $*$ called fusion, and a (relevant) implication $\Rightarrow$. The operation $*$ can be considered as a genuine modal operator of necessity. So, an extension of the Classical Propositional Calculus $\mathcal{PC}$ by means of an operator of fusion $*$ can be seen as a particular polymodal logic. So, the model theory of this type of logics is a particular case of the model theory for $\mathcal{ML}$. But the implication $\Rightarrow$ is not a modal operator, and consequently the results of model theory develop for $\mathcal{ML}$ are not directly applicable to logics with this kind of implication. Thus, it is natural to ask if the questions typical of model theory of $\mathcal{ML}$ remain valid for some Boolean relevant logics. We address some of these questions in the present work. If we like to prove results on the model theory of $\mathcal{BR}$ following the lines of the results for $\mathcal{ML}$, we shall need an adequate notion of bisimulation. In [9] G. Restall introduces a notion of bisimulation for some substructural logics, including some relevant logics. In this paper we will apply this notion of bisimulation to the Boolean relevant logic $\mathcal{BR}$.

In Section 2 we recall the necessary definitions and notions for the develop of this paper. In Section 3 we define bisimulations and we show a characterization of bisimulation in terms of p-morphisms. In Section 4 we introduce the class of m-saturated models. In general, the equivalence between two models does not always imply bisimilarity. We prove that if the models are m-saturated then equivalence implies bisimilarity. We also introduce the class of image-finite models and we prove that an image-finite model is logically bisimilar to any other model. This result enables to us prove that the class of models bisimilar to one image-finite model is contained in any class of models where equivalence implies bisimilarity. In Section 5 we define the ultrafilter extension of a model. The ultrafilter extension of a model is an m-saturated model that can view as a kind of completation of the original model. The main result of this section is the characterization of the equivalence between two models as a bisimulation between the associated ultrafilter extensions.

2 Preliminaries

We shall consider a language $\mathcal{L}$ with a denumerable set of variables $Var = \{p_0, p_1, ..., ...\}$, the binary connectives $\vee, \rightarrow, \circ$, the unary connective $\neg$ and the propositional constants t and $\top$. The connectives $\wedge$, $\supset$ and the constant $\bot$ are defined as follows: $p \wedge q = \neg(\neg p \vee \neg q)$, $p \supset q = \neg p \vee q$ and $\bot = \neg\top$. The set of formulas Fm is defined by the rules:

$$\varphi ::= p \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi \circ \psi \mid \neg\varphi \mid t \mid \top,$$

where $p \in Var$ and $\varphi, \psi \in Fm$.

Definition 1 A *classical relevant frame, or frame for short,* is a relational structure $\mathcal{F} = \langle X, T, E \rangle$, *where* $E \subseteq X$, T is a ternary relation defined on X and satisfies the condition: $\forall x, y \in X \quad (x = y \Leftrightarrow \exists e \in E$ such that $(e, x, y) \in T)$..

Let $\mathcal{F}$ be a frame. We write for $x, y, z \in X$, $(x, y, z) \in T \Leftrightarrow (x, y) \in T^{-1}(z) \Leftrightarrow (y, z) \in T(x)$. The set of all subsets of X will be denoted by $\mathcal{P}(X)$. The set of finite subsets of X will be denoted by $\mathcal{P}_f(X)$. The complement of a set $Y \subseteq X$ will be symbolized by Y^c.

A *valuation* is a function $V : Var \rightarrow \mathcal{P}(X)$. Any valuation V can be extended to the set Fm as follows:

1. $V(\top) = X$
2. $V(t) = E$
3. $V(\neg\varphi) = V(\varphi)^c$
4. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$
5. $V(\varphi \circ \psi) = \{z \in X : \exists x, y : (x, y, z) \in T \,\&\, x \in V(\varphi) \,\&\, y \in V(\psi)\}$
6. $V(\varphi \rightarrow \psi) = \{x \in X : \forall y, z : (x, y, z) \in T \,\&\, y \in V(\varphi)$, then $z \in V(\psi)\}$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a frame and V is a valuation on $\mathcal{F}$. Now, we shall define the notions of vality on models and frames. Let $\mathcal{F}$ be a frame and let V be a valuation on it. Let $\varphi \in Fm$ and $x \in X$. The formula φ is *valid* in $\mathcal{M} = \langle \mathcal{F}, V \rangle$ at

x, in symbols $\mathcal{M} \vDash_x \varphi$, if $x \in V(\varphi)$. The formula φ is *valid in* $\mathcal{M}$, in symbols $\mathcal{M} \vDash \varphi$, if $E \subseteq V(\varphi)$. The formula φ is valid in $\mathcal{F}$, in symbols $\mathcal{F} \vDash \varphi$, if for any valuation V defined on $\mathcal{F}$, $\langle \mathcal{F}, V \rangle \vDash \varphi$.

Let $\mathbf{K}$ be the class of all frames. The Boolean relevant logic $\mathcal{BR}$ can be defined as the logic in the language $\mathcal{L}$ generated by the class $\mathbf{K}$, i.e., $\mathcal{BR} = \{\varphi \in Fm : \mathcal{F} \vDash \varphi$ for all $\mathcal{F} \in \mathbf{K}\}$.. An axiomatization of the logic $\mathcal{BR}$ can be obtained taking the axioms given in [12] for the relevant logic $\mathcal{R}$ plus any axiomatization of the Classical Propositional Calculus.

A *Boolean relevant algebra*, or BR-algebra, is an algebra $\langle A, \vee, \rightarrow, \circ, \neg, e, 1\rangle$ such that $\langle A, \vee, \neg, 1\rangle$ is a Boolean algebra and

1. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$
2. $(b \vee c) \circ a = (b \circ c) \vee (c \circ a)$
3. $a \circ 0 = 0$
4. $e \circ a = a$
5. $a \circ b \leq c \Leftrightarrow a \leq b \rightarrow c.$

Let A be a BR-algebra. The set of all ultrafilters of A is denoted by $Ul(A)$. The set of all filters of A is symbolized by $F_i(A)$. Let $F, H \in F_i(A)$. Define the set

$$F \circ H = \{x \in A : f \circ h \leq x \text{ for some pair } (f, g) \in F \times H\}.$$

It is easy to see that $F \circ H$ is a filter of A. In the set $Ul(A)$ define the ternary relation T_A by:

$$(P, Q, D) \in T_A \Leftrightarrow P \circ Q \subseteq P$$

Let $E(A) = \{P \in Ul(A) : e \in P\}$. By the results of A. Urquhart [12] (see also [3]), the relational structure $\mathcal{F}(A) = \langle Ul(A), T_A, E(A)\rangle$ is a frame, called the associated frame to A.

The following results summarize known results on the relation T_A. For a proof of these results the reader is referred to [3], [10] or [12].

Theorem 2 *Let A be a BR-algebra. Then:*

1. *Let $F_1, F_2 \in F_i(A)$ and $P \in Ul(A)$. If $F_1 \circ F_2 \subseteq P$, then there exist $Q, D \in Ul(A)$ such that $F_1 \subseteq Q$, $F_2 \subseteq D$ and $Q \circ D \subseteq P$.*
2. *Let $a, b \in A$ and $P \in Ul(A)$. Then $a \circ b \in P$ if and only if there exit $Q, D \in Ul(A)$ such that $a \in Q$, $b \in D$ and $Q \circ D \subseteq P$.*
3. *Let $a, b \in A$ and $P \in Ul(A)$. Then $a \rightarrow b \in P$ if and only if for any $Q, D \in Ul(A)$ such that, if $P \circ Q \subseteq D$ and $a \in Q$, then $b \in D$.*
4. *For all P, $Q \in Ul(A)$ and for all $E \in E(A)$, $E \circ P \subseteq Q$ if and only if $P = Q$.*

All frame $\mathcal{F}$ has associated a BR-algebra. On the Boolean algebra $\langle \mathcal{P}(X), \cup, ^c, X\rangle$, where ^{c} is the set complement, we define the operations $*$ and $\Rightarrow$ as follows:

$$U * V = \left\{x \in X : T^{-1}(x) \cap (U \times V)\right\},$$

$$U \Rightarrow V = \{x \in X : T(x) \cap (U \times V^c) = \emptyset\},$$

for all $U, V \in \mathcal{P}(X)$. It is easy to check that the structure $A(\mathcal{F}) = \langle \mathcal{P}(X), \cup, *, \Rightarrow, ^c, E, X\rangle$ is a BR-algebra.

Remark. We note that if $\mathcal{M} = \langle \mathcal{F}, V\rangle$ is a model, then the set $D_V = \{V(\varphi) : \varphi \in Fm\}$ is a BR-subalgebra of the BR-algebra $A(\mathcal{F})$, where $V(\varphi \circ \psi) = V(\varphi) * V(\psi)$, $V(\varphi \to \psi) = V(\varphi) \Rightarrow V(\psi)$, and $V(t) = E$. This fact we will use without mention in the rest of this paper.

Let X_c be the $\mathcal{L}$-maximal consistent theories. The *canonical frame* is the structure $\mathcal{F}_c = \langle X_c, T_c, E_c\rangle$ where $(P, Q, D) \in T_c$ if and only if $P \circ Q \subseteq D$, and $E_c = \{P \in X_c : t \in P\}$. The *canonical model* is the model $\mathcal{M}_c = \langle \mathcal{F}_c, V_c\rangle$ where V_c is the valuation defined by $V_c(p) = \{P \in X_c : p \in P\}$, for $p \in Var$.

3 Bisimulations

In this section we shall define the classical relevant bisimulations. In classical modal logic the notion of bisimulation is an important tool for establish an equivalence relation between pointed models. Bisimulations are also know as strong bisimulations or zigzag relations. We prove that two bisimilar models are the same theory, and that the notions of p-morphism is a particular case of the notion of bisimulation.

Definition 3 Let $\mathcal{F}_1 = \langle X_1, T_1, E_1\rangle$ and $\mathcal{F}_2 = \langle X_2, T_2, E_2\rangle$ be two frames. A relation $B \subseteq X_1 \times X_2$ is a *bisimulation* between $\mathcal{F}_1$ and $\mathcal{F}_2$ if:

B0. If $(a, b) \in B$, then $a \in E_1$ iff $b \in E_2$.

B1. If $(a, b) \in B$ and $(a, x, y) \in T_1$, then there exist $x', y' \in X_2$ such that $(b, x', y') \in T_2$, $(x, x') \in B$ and $(y, y') \in B$.

B2. If $(a, b) \in B$ and $(x, y, a) \in T_1$, then there exist $x', y' \in X_2$ such that $(x', y', b) \in T_2$ and $(x, x') \in B$ and $(y, y') \in B$.

B3. If $(a, b) \in B$ and $(b, x', y') \in T_2$, then there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$ and $(x, x') \in B$ and $(y, y') \in B$.

B4. *If* $(a, b) \in B$ and $(x', y', b) \in T_2$, then there exist $x, y \in X_1$ such that $(x, y, a) \in T_1$, $(x, x') \in B$ and $(y, y') \in B$.

Let $\mathcal{M}_1 = \langle \mathcal{F}_1, V_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2\rangle$ be two models. A *bisimulation* B between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a bisimulation between the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ such that:

V. If $(a, b) \in B$, then $a \in V_1(p)$ iff $b \in V_2(p)$, for any $p \in Var$.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. Let $x \in X_1$ and $y \in X_2$. We shall say that x and y are *bisimilar* if there exist a bisimulation B between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $(x, y) \in B$. In this case, we shall write $\mathcal{M}_1, x \underline{\leftrightarrow} \mathcal{M}_2, y$. A bisimulation is said *total* if $domB = X_1$ and $rangB = X_2$. The models $\mathcal{M}_1$ and $\mathcal{M}_2$ are *bisimilars*, in symbols $\mathcal{M}_1 \underline{\leftrightarrow} \mathcal{M}_2$, if there exists a total bisimulation B between them.

Let $\mathcal{M}$ be a model. For any $x \in \mathcal{M}$, we define the set $F_x^{\mathcal{M}} = \{\varphi \in Fm : x \in V(\varphi)\}$. We note that for each $x \in \mathcal{M}$, the set $F_x^{\mathcal{M}}$ is a maximal and consistent theory.

Definition 4 Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. Let $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$. We shall say that a and b are *equivalent*, in symbols $a \approx b$, if $F_a^{\mathcal{M}_1} = F_b^{\mathcal{M}_2}$.

The next result said that all bisimulation B between two models $\mathcal{M}_1$ and $\mathcal{M}_2$ is contained in the relation $\approx$. This result was proved by G. Restall [9] but we shall give there a proof for completeness.

Lemma 5 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models and let B a bisimulation between them. Then for any $a \in X_1$ and for any $b \in X_2$, if $(a, b) \in B$, then $F_a^{\mathcal{M}_1} = F_b^{\mathcal{M}_2}$.*

Proof. By induction on the complexity of the formulas. We only prove the cases for formulas $\varphi \circ \psi$ and $\varphi \to \psi$. Let $(a, b) \in B$ and $a \in V_1(\varphi \circ \psi)$. Then, there exists $x, y \in X_1$ such that $(x, y, a) \in T_1$, $x \in V_1(\varphi)$ and $y \in V_1(\psi)$. By clause **B2**, there exist $x', y' \in X_2$ such that $(x', y', b) \in T_2$ such that $(x, x') \in B$ and $(y, y') \in B$. By inductive hypothesis , $x' \in V_2(\varphi)$ and $y' \in V_2(\psi)$. Then, $b \in V_2(\varphi \circ \psi)$.

Let $(a, b) \in B$ and $a \in V_1(\varphi \to \psi)$. We shall prove that $b \in V_2(\varphi \to \psi)$. Let $x', y' \in X_2$ such that $(b, x', y') \in T_2$ and $x' \in V_2(\varphi)$. By clause **B3**, there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$ such that $(x, x') \in B$ and $(y, y') \in B$. By inductive hypothesis, $x \in V_1(\varphi)$, and since $a \in V_1(\varphi \to \psi)$, we get that $y \in V_1(\psi)$. Again by inductive hypothesis $y' \in V_2(\psi)$. Thus $b \in V_2(\varphi \to \psi)$. The cases when $b \in V_2(\varphi \circ \psi)$ and $b \in V_2(\varphi \to \psi)$ are analyzed similarly, using the clauses **B1** and **B4**, respectively. ■

We shall investigate the relation between morphisms of models and bisimulations between models. The following definition is an adaptation of the definition given by A. Urquarth [12].

Definition 6 Let $\langle \mathcal{F}_1, V_1 \rangle$ and $\langle \mathcal{F}_2, V_2 \rangle$ two models. A function $f : X_1 \to X$, is a *p-morphisms* between the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ if it verifies:

P0. $f^{-1}(E_1) \subseteq E_0$.

P1. If $(x, y, a) \in T_1$, implies that $(f(x), f(y), f(a)) \in T_2$.

P2. If $(f(a), x', y') \in T_2$, then there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $f(x) = x'$, and $f(y) = y'$.

P3. If $(x', y', f(a)) \in T_2$, then there exist $x, y \in X_1$ such that $(x, y, a) \in T_1$, $f(x) = x'$ and $f(y) = y'$.

A function $f : X_1 \to X_2$ is a p-morphisms between the models $\langle \mathcal{F}_1, V_1 \rangle$ and $\langle \mathcal{F}_2, V_2 \rangle$ if is a p-morphism between the frames $\mathcal{F}_1$ and $\mathcal{F}_2$, and verifies:

P4. For all $p \in Var$, $V_1(p) = f^{-1}(V_2(p))$.

In the next result we give an equivalent definition of bisimulation.

Lemma 7 *Let $\mathcal{F}_1 = \langle X_1, T_1, E_1\rangle$ and $\mathcal{F}_2 = \langle X_2, T_2, E_2\rangle$ be two frames. Let $B \subseteq X_1 \times X_2$. Then the following conditions are equivalents:*

1. *B is a bisimulation between $\mathcal{F}_1$ and $\mathcal{F}_2$.*
2. *There exists a ternary relation $T_B \subseteq B^3$ and $E_B \subseteq B$ such that $\langle B, T_B, E_B\rangle$ is a frame and the projections $\pi_1 : B \to X_1$ and $\pi_2 : B \to X_2$ are p-morphisms.*

Proof. $1 \Rightarrow 2$. Let us define the subset E_B of B, and the subset T_B of B^3 by $E_B = E_1 \times E_2$ and

$$((x_1, x_2), (y_1, y_2), (z_1, z_2)) \in T_B \Leftrightarrow (x_1, y_1, z_1) \in T_1 \text{ and } (x_2, y_2, z_2) \in T_2,$$

respectively. The proof of condition 3. of Definition 3 is easy and left to reader. Thus, $\langle B, T_B, E_B\rangle$ is a frame. We prove that $\pi_1 : B \to X_1$ is a p-morphism. We prove P0, i.e., $\pi_1^{-1}(E_1) \subseteq E_B$. Let $(x, y) \in \pi_1^{-1}(E_1)$. then, $\pi_1(x, y) = x \in E_1$. By condition $B0$, $y \in E_2$. So, $(x, y) \in E_B$.

We prove P2. Let $(\pi_1(x_1, x_2), y_1, z_1) \in T_1$. Then, $(x_1, x_2) \in B$ and $(x_1, y_1, z_1) \in T_1$. By the condition B1 we have that there exist $y_2, z_2 \in X_2$ such that $(x_2, y_2, z_2) \in T_2$, $(y_1, y_2) \in B$ and $(z_1, z_2) \in B$. Then, $(x_2, y_2, z_2) \in T_2$, $\pi_1(y_1, y_2) = y_1$, and $\pi_1(z_1, z_2) = z_1$. The proof of P3 is similar.

$2 \Rightarrow 1$. We prove the condition B2. Let $x_1, y_1, z_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \in B$ and $(x_1, y_1, z_1) \in T_1$. Since, $\pi_1(x_1, x_2) = x_1$, we get $(\pi_1(x_1, x_2), y_1, z_1) \in T_1$. As π_1 is a p-morphims, there exists (y_1, y_2), $(z_1, z_2) \in B$ such that $((x_1, x_2), (y_1, y_2), (z_1, z_2)) \in T_B$ and $\pi_1(y_1, y_2) = y_1$ and $\pi_1(z_1, z_2) = z_1$. Then, $(x_2, y_2, z_2) \in T_2$ and $(y_1, y_2) \in B$, and $(z_1, z_2) \in B$. The other conditions are proved similarly. ■

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ two frames. Let us consider a function $f : X_1 \to X_2$. We define a relation $B_f \subseteq X_1 \times X_2$ as follows:

$$(a, b) \in B_f \Leftrightarrow f(a) = b.$$

Proposition 8 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ two frames and a function $f : X_1 \to X_2$. Let us consider the relation B_f defined above. Then*

1. *B_f verifies* B1 *or* B2 *iff f verifies* P1.
2. *B_f verifies* B3 *iff f verifies* P2.
3. *B_f verifies* B4 *iff f verifies* P3.

Proof. 1. Assume that B_f verifies $B2$. Let $(x, y, a) \in T_1$ and let us consider $b = f(a)$. So $(a, b) \in B_f$. Then there exists $x', y' \in X_2$ such that $(x', y', b) \in T_2$, $f(x) = x'$ and $f(y) = y'$. Then $(f(x), f(y), f(a)) \in T$.

The proof in the other direction is immediate.

2. Suppose now that f verifies P2. Let $f(a) = b$ and $(b, x', y') \in T_2$. Then $(f(a), x', y') \in T_2$ implies that there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $f(x) = x'$ and $f(y) = y'$. Thus, B3 is valid.

Suppose that B_f verifies B3 and let $(f(a), x', y') \in T_2$. Let $b = f(a)$. Then there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $f(x) = x'$ and $f(y) = y'$. So, P2 is valid.

The assertion 3. is analyzed similarly and left to the reader. ■

The theory of a model $\langle \mathcal{F}_1, V_1 \rangle$ is the set $Th(\langle \mathcal{F}_2, V_2 \rangle) = \{\varphi \in Fm : \langle \mathcal{F}_1, V_1 \rangle \vDash \varphi\}$. We shall say that two models $\langle \mathcal{F}_1, V_1 \rangle$ and $\langle \mathcal{F}_2, V_2 \rangle$ are equivalents if $Th(\langle \mathcal{F}_1, V_1 \rangle) = Th(\langle \mathcal{F}_2, V_2 \rangle)$.

Corollary 9 *A function $f : \langle \mathcal{F}_1, V_1 \rangle \to \langle \mathcal{F}_2, V_2 \rangle$ is a p-morphism iff the relation B_f is a bisimulation. Thus, for any $x \in X_1$, a and $f(a)$ are bisimilares. If f is surjective then $Th(\langle \mathcal{F}_1, V_1 \rangle) = Th(\langle \mathcal{F}_2, V_2 \rangle)$.*

Proof. It is immediate by the above results. ■

4 M-saturated models

It is well know that the converse of the Lemma 5 does not hold in general, because two points may be equivalent without beging bisimilar. This fact brings about the following cuestions. When two bisimilar points are equivalents. In other words, when the relation $\approx$ is a bisimulation. In this section we shall introduce the class of models that have the property of that for two models $\mathcal{M}_1$ and $\mathcal{M}_2$ the relation $\approx$ between them is a bisimulation. The principal class of models studied in this section is the class of m-saturated models. The notion of m-saturation is well know in classical modal logic (see [4] and [6]).

Recall that a set of formulas Γ is said *satisfiable* in a model $\mathcal{M}$ if there exists $x \in X$ such that $\mathcal{M} \vDash_x \varphi$, for all $\varphi \in \Gamma$. The set Γ is *finitely satisfiable* in $\mathcal{M}$ if every finite subset of Γ is satisfiable in $\mathcal{M}$. For any subset $\Gamma \subseteq Fm$ we shall write $V(\Gamma) = \bigcap_{\varphi \in \Gamma} V(\varphi)$ and $V(\neg\Gamma) = \bigcap_{\varphi \in \Gamma} V(\neg\varphi)$.

Definition 10 Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model. We shall say that $\mathcal{M}$ is *m-saturated* if for all sets $\Gamma, \Delta \subseteq Fm$ and for all $a \in X$ we have:

M1. If $T^{-1}(a) \cap (V(\Gamma_0) \times V(\Delta_0)) \neq \emptyset$ for all $\Gamma_0 \in \mathcal{P}_f(\Gamma)$ and for all $\Delta_0 \in \mathcal{P}_f(\Delta)$, then it holds

$$T^{-1}(a) \cap (V(\Gamma) \times V(\Delta)) \neq \emptyset.$$

M2. If $T(a) \cap (V(\Gamma_0) \times V(\neg\Delta_0)) \neq \emptyset$ for all $\Gamma_0 \in \mathcal{P}_f(\Gamma)$ and for all $\Delta_0 \in \mathcal{P}_f(\Delta)$, then it holds

$$T(a) \cap (V(\Gamma) \times V(\neg\Delta)) \neq \emptyset.$$

Definition 11 A class of models $\mathbf{K}$ is called a *Hennessey-Milner class* if for $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{K}$, and for all $x \in X_1$, $y \in X_2$, if $x \approx y$, then $\mathcal{M}_1, x \underline{\leftrightarrow} \mathcal{M}_2, y$. That is, the equivalence $\approx$ is a bisimulation. The model $\mathcal{M}$ is said that has the *Hennessy-Milner property* if $\approx$ is a bisimulation on it.

For classical modal logic the definition above is given in [4] and [6]. The most know example of Hennessey-Milner class is the class **Imf** of all image-finite models.

Definition 12 A model $\mathcal{M}$ is *image-finite* if for all $x \in X$, $T(x)$ and $T^{-1}(x) \in \mathcal{P}_f(X \times X)$.

We note that this definition is not exactly to the definition given in [9].

Lemma 13 *All image-finite model $\mathcal{M}$ is m-saturated. Thus, the class* **Imf** *is a Hennessy-Milner class.*

Proof. Let $\mathcal{M} = \langle X, T, E, V\rangle$ be a image-finite model. We only to check the condition **M2**. Let $\Gamma, \Delta \subseteq Fm$, and $a \in X$ such that for any pair $(\Gamma_0, \Delta_0) \in \mathcal{P}_f(\Gamma) \times \mathcal{P}_f(\Delta)$, $T(a) \cap (V(\Gamma_0) \times V(\neg\Delta_0)) \neq \emptyset$. Since $T(a)$ is a finite subset of $X \times X$, then let $T(a) = \{(x_1, y_1), ..., (x_n, y_n)\}$. Suppose that

$$T(a) \cap (V(\Gamma) \times V(\neg\Delta)) = \emptyset.$$

Then for any $i \in \{1, ..., n\}$, $x_i \notin V(\Gamma)$ or $y_i \notin V(\neg\Delta)$, i.e., for each $i \in \{1, ..., n\}$ there exists a pair $(\varphi_i, \psi_i) \in \Gamma \times \Delta$ such that $(x_i, y_i) \notin (V(\varphi_i) \times V(\neg\psi_i))$. Let us consider the sets $\Gamma_0 = \{\varphi_1, ..., \varphi_n\}$ and $\Delta_0 = \{\psi_1, ..., \psi_n\}$. By assumption, $T(a) \cap (V(\Gamma_0) \times V(\neg\Delta_0)) \neq \emptyset$, i.e., there exists some $i \in \{1, ..., n\}$ such that $(x_i, y_i) \in (V(\Gamma_0), V(\neg\Delta))$, which is a contradiction. Thus, M2 is valid. The proof of the condition M1 is similar and left to the reader. ■

Theorem 14 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two m-saturated models. Let $a \in X_1$ and $b \in X_2$. Let us consider the relation $(a, b) \in B \Leftrightarrow a \approx b$. Then B is a bisimulation.*

Proof. Let $a \in X_1$ and $b \in X_2$ such that $F_a^{\mathcal{M}_1} = F_b^{\mathcal{M}_2}$. It is clear that the conditions B0 and V are satisfied. We shall the check the conditions B2 and B3. The proof of the conditions B1 and B4 are similars and left to the reader.

B2. Let $(x, y, a) \in T_1$. Let us consider the sets $F_x^{\mathcal{M}_1}$ and $F_y^{\mathcal{M}_1}$. Since

$$F_x^{\mathcal{M}_1} \circ F_y^{\mathcal{M}_1} = \{\varphi \circ \psi : \varphi \in F_x^{\mathcal{M}_1} \text{ and } \psi \in F_y^{\mathcal{M}_1}\} \subseteq F_a^{\mathcal{M}_1} = F_b^{\mathcal{M}_2},$$

then for all $(\varphi, \psi) \in F_x^{\mathcal{M}_1} \times F_y^{\mathcal{M}_1}$, $b \in V_2(\varphi \circ \psi)$, i.e., $T_2^{-1}(b) \cap (V_2(\varphi) \times V_2(\psi)) \neq \emptyset$. So, for all $\Gamma_0 \in \mathcal{P}_f(F_x^{\mathcal{M}_1})$ and for all $\Delta_0 \in \mathcal{P}_f(F_y^{\mathcal{M}_1})$ we get,

$$T_2^{-1}(b) \cap (V_2(\Gamma_0) \times V_2(\Delta_0)) \neq \emptyset.$$

Since $\mathcal{M}_2$ is m-saturated, $T_2^{-1}(b) \cap (V_2(F_x^{\mathcal{M}_1}) \times V_2(F_y^{\mathcal{M}_1})) \neq \emptyset$. Then there exists $x', y' \in x_2$ such that $(x', y', b) \in T_2$, $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$.

B3. Let $(b, x', y') \in T_2$. Then, $F_b^{\mathcal{M}_2} \circ F_{x'}^{\mathcal{M}_2} \subseteq F_{y'}^{\mathcal{M}_2}$. Let $\varphi \in F_{x'}^{\mathcal{M}_2}$ and $\neg\psi \in F_{y'}^{\mathcal{M}_2}$. Then, $b \notin V_2(\varphi \to \psi)$. This implies that $a \notin V_1(\varphi \to \psi)$. It follows that there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $x \in V_1(\varphi)$ and $y \in V_1(\neg\psi)$. Thus, for all $\Gamma_0 \in \mathcal{P}_f(F_{x'}^{\mathcal{M}_2})$ and for all $\Delta_0 \in \mathcal{P}_f(F_{y'}^{\mathcal{M}_2})$ we have $T_1(a) \cap (V_1(\Gamma_0) \times V_1(\neg\Delta_0)) \neq \emptyset$. Since $\mathcal{M}_1$ is m-saturated,

$$T_1(a) \cap (V_1(F_{x'}^{\mathcal{M}_1}) \times V_1(\neg F_{y'}^{\mathcal{M}_2})) \neq \emptyset.$$

Then there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $F_{x'}^{\mathcal{M}_2} = F_x^{\mathcal{M}_1}$ and $F_{y'}^{\mathcal{M}_2} = F_y^{\mathcal{M}_1}$. ■

Corollary 15 *The class* **Sat** *of all m-saturated models and the class* **Imf** *all image-finite models has the Hennessey-Milner property.*

A very important property of the class **Imf** is given in the next result.

Theorem 16 *All image-finite model is logically bisimilar to any other model.*

Proof. Let $\mathcal{M}_1 = \langle X_1, T_1, E_1, V_1 \rangle$ be an image-finite model. Let $\mathcal{M}_2 = \langle X_2, T_2, E_2, V_2 \rangle$ be any other model. We prove that the relation $\approx$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a bisimulation. Let $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$ such that $a \approx b$.

B3. Let $(b, x', y') \in T_2$. Let us consider the sets $F_{x'}^{\mathcal{M}_2}$ and $F_{y'}^{\mathcal{M}_2}$ and let us consider finite subsets

$$\Gamma_0 = \{\varphi_1, ..., \varphi_n\} \subseteq F_{x'}^{\mathcal{M}_2} \text{ and } \Delta_0 = \{\psi_1, ..., \psi_k\} \subseteq F_{y'}^{\mathcal{M}_2}.$$

Let $\varphi = \varphi_1 \wedge ... \wedge \varphi_n$ and $\psi = \psi_1 \vee ... \vee \psi_k$. Then, $x' \in V_2(\varphi)$ and $y' \in V_2(\psi)$. This implies that $T_2(b) \cap (V_2(\varphi), V_2(\neg(\neg\psi))) \neq \emptyset$, i.e., $b \notin V_2(\varphi \to \neg\psi)$. Since $a \approx b$, $a \notin V_1(\varphi \to \neg\psi)$. It follows that there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $x \in V_1(\varphi)$ and $y \in V_1(\psi)$. Then for any $\Gamma_0 \in \mathcal{P}_f(F_{x'}^{\mathcal{M}_2})$ and for any $\Delta_0 \in \mathcal{P}_f(F_{y'}^{\mathcal{M}_2})$, we get $T_1(a) \cap (V_1(\Gamma_0) \times V_1(\Delta_0)) \neq \emptyset$. As $\mathcal{M}_1$ is image-finite, then by Lemma 13 $\mathcal{M}_1$ is m-saturated. It follows that

$$T_1(a) \cap \left(V_1\left(F_{x'}^{\mathcal{M}_2}\right) \times V_1\left(F_{y'}^{\mathcal{M}_2}\right)\right) \neq \emptyset.$$

This means that there exist $x, y \in X_1$ such that $(a, x, y) \in T_1$, $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$.

B4. Let $(x', y', b) \in T_2$. As above, it is easy to prove that

$$T_1^{-1}(a) \cap \left(V_1\left(F_{x'}^{\mathcal{M}_2}\right) \times V_1\left(\neg F_{y'}^{\mathcal{M}_2}\right)\right) \neq \emptyset.$$

So, there exist $x, y \in X_1$ such that $(x, y, a) \in T_1$, $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$.

B2. Let $(x, y, a) \in T_1$. Since $\mathcal{M}_1$ is image-finite, then $T_1^{-1}(a) = \{(x_1, y_1), ..., (x_n, y_n)\}$. We prove that for each $1 \leq i \leq n$ there exists a pair of formulas (ϕ_i, φ_i) (called the characteristic pair of (x_i, y_i)) such that for all $1 \leq j \leq n$,

$$x_i \approx x_j \Leftrightarrow x_j \in V_1(\phi_i) \text{ and } y_i \approx y_j \Leftrightarrow y_j \in V_1(\varphi_i).$$

Indeed. For each $i \neq j$ such that $x_i \not\approx x_j$ there exist $\phi_{ij} \in Fm$ such that $x_i \in V_1(\phi_{ij})$ and $x_j \notin V_i(\phi_{ij})$. Then $x_i \in V_1\left(\bigwedge_{i \neq j} \phi_{ij}\right) = V_1(\phi_i)$ and $x_j \notin V_1(\phi_i)$ for all $x_i \not\approx x_j$. Then, $x_i \approx x_j \Leftrightarrow x_j \in V_1(\phi_i)$. Similarly, there exists $\varphi_i \in Fm$ such that $y_i \approx y_j \Leftrightarrow y_j \in V_1(\varphi_i)$.

Let us (φ_x, φ_y) be the characteristic pair of (x, y). Then $a \in V_1(\varphi_x \circ \varphi_y)$. Since $a \approx b$, $b \in V_2(\varphi_x \circ \varphi_y)$. Consequently there exist $x', y' \in X_2$ such that $(x', y', b) \in T_2$, $x' \in V_2(\varphi_x)$ and $y' \in V_2(\varphi_y)$. We prove that $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$. Let α_1, $\alpha_2 \in Fm$ such that $x' \in V_2(\alpha_1)$ and $y' \in V_2(\alpha_2)$. Since, $x' \in V_2(\alpha_1 \wedge \varphi_x)$ and $y' \in V_2(\alpha_2 \wedge \varphi_y)$, $b \in V_2((\alpha_1 \wedge \varphi_x) \circ (\alpha_2 \wedge \varphi_y))$. So, $a \in V_1((\alpha_1 \wedge \varphi_x) \circ (\alpha_2 \wedge \varphi_y))$. Then, there exist $k_1, k_2 \in X_1$ such that $k_1 \in V_1(\alpha_1 \wedge \varphi_x)$, $k_2 \in V_1(\alpha_2 \wedge \varphi_y)$ and $(k_1, k_2, a) \in T_1$. But this implies that $k_1 \approx x$ and $k_2 \approx y$. Thus, $x \in V_1(\alpha_1)$ and $y \in V_2(\alpha_2)$. We conclude that $x \approx x'$ and $y \approx y'$.

B1. Let $(a,x,y) \in T_1$. As above, let (φ_x, φ_y) be the characteristic pair of (x,y). So, $x \in V_1(\varphi_x)$ and $y \in V_1(\varphi_y)$ .Then $a \notin V_1(\varphi_x \to \neg\varphi_y)$.Since $a \approx b$, $b \notin V_2(\varphi_x \to \neg\varphi_y)$. This implies that there exist $x', y' \in X_2$ such that $(b, x', y') \in T_2$, $x' \in V_2(\varphi_x)$ and $y' \in V_2(\varphi_y)$. We prove that $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$. Let α_1, $\alpha_2 \in Fm$ such that $x' \in V_2(\alpha_1)$ and $y' \in V_2(\alpha_2)$. Then, since $b \notin V_2((\alpha_1 \wedge \varphi_x) \to \neg(\alpha_2 \wedge \varphi_y))$, $a \notin V_1((\alpha_1 \wedge \varphi_x) \to \neg(\alpha_2 \wedge \varphi_y))$. So, there exist $k_1, k_2 \in X_1$ such that $k_1 \in V_1(\alpha_1 \wedge \varphi_x)$, $k_2 \in V_1(\alpha_2 \wedge \varphi_y)$ and $(a, k_1, k_2) \in T_1$. It follows that $F_x^{\mathcal{M}_1} = F_{x'}^{\mathcal{M}_2}$ and $F_y^{\mathcal{M}_1} = F_{y'}^{\mathcal{M}_2}$.

Thus, $\approx$ is a bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. ■

Let **Int** be the intersection of all maximal Hennessy-Milner classes.

Corollary 17 $B(\mathbf{Imf}) = \{\mathcal{N} : \exists \mathcal{M} \in \mathbf{Imf} : \mathcal{M} \underline{\leftrightarrow} \mathcal{N}\} \subseteq \mathbf{Int}$.

Proof. Let $\mathcal{M} \in B(\mathbf{Imf})$. Then there exists an image-finite model $\mathcal{M}'$ such that $\mathcal{M}$ and $\mathcal{M}'$ are bisimilars. Let $\mathbf{K}$ be any maximal Hennessy-Milner class. By above theorem, $\mathcal{M}'$ is bisimilar to any model of $\mathbf{K}$. So, $\mathbf{K} \cup \{\mathcal{M}'\}$ is a Hennessey-Milner class. Since $\mathbf{K}$ is maximal, $\mathcal{M}' \in \mathbf{K}$. Hence, $\mathbf{K} = B(\mathbf{K})$ and $\mathcal{M} \in \mathbf{K}$. Thus, $\mathcal{M} \in \mathbf{Int}$, and this implies that $B(\mathbf{Imf}) \subseteq \mathbf{Int}$. ■

Problem: When $\mathbf{Int} \subseteq B(\mathbf{Imf})$?

5 Ultrafilter extension of a model

In this section we define the ultrafilter extension of a model. This construction is important because give a characterization of the equivalence between models by means of ultrafilter extensions.

Let $\mathcal{F}$ be a frame. The *ultrafilter extension of* $\mathcal{F}$ is the frame

$$U_e(\mathcal{F}) = \langle\, Ul(\mathcal{P}(X)), T_u, E_u \rangle,$$

where

- $Ul(\mathcal{P}(X))$ is the set of all ultrafilters on X,
- T_u is a ternary relation defined on $Ul(\mathcal{P}(X))$ by:

$$(P, Q, D) \in T_u \Leftrightarrow P * Q \subseteq D,$$

  where $P*Q = \{U * V : U \in P, V \in Q\}$ and $U*V = \{x \in X : T^{-1}(x) \cap (U \times V) \neq \emptyset\}$.

- $E_u = \{P \in Ul(\mathcal{P}(X)) : E \in P\}$.

In other words, the ultrafilter extension of $\mathcal{F}$ is the frame associated to classical relevant algebra $\mathcal{P}(X)$.

Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be a model. The *ultrafilter extension* of $\mathcal{M}$ is the model

$$U_e(\mathcal{M}) = \langle U_e(\mathcal{F}), V_u\rangle$$

where the valuation V_u is defined by

$$V_u(P) = \{P \in Ul(\mathcal{P}(X)) : V(p) \in P\},$$

for any $p \in Var$.

Given an element x of a set X, it is easy to see that the collection

$$H(x) = \{U \subseteq X : x \in U\}$$

is an ultrafilter of $\mathcal{P}(X)$, called the *principal ultrafilter generated by* x. A important property the ultrafilter extension of a model is that is m-saturated.

Theorem 18 *Let $\mathcal{M}$ be a model. Then*

1. *For any $\varphi \in Fm$, $V_u(\varphi) = \{P \in Ul(\mathcal{P}(X)) : V(\varphi) \in P\}$.*
2. *For any $x \in X$, $x \approx H(x)$.*

Proof. 1. The proof is by induction on the complexity of the formulas. We consider only the case $\varphi \to \psi$. The other cases are analyzed similarly and left to the reader.

Let $P \in Ul(\mathcal{P}(X))$ and $\varphi, \psi \in Fm$ such that $V(\varphi \to \psi) \in P$. We prove that $P \in V_u(\varphi \to \psi)$. Let $Q, D \in Ul(\mathcal{P}(X))$ such that $(P, Q, D) \in T_u$ and $Q \in V_u(\varphi)$. By inductive hypothesis, $V(\varphi) \in Q$, and since $V(\varphi \to \psi) \in P$, then we get $V(\psi) \in D$. Again by inductive hypothesis, $D \in V_u(\psi)$. Thus, $P \in V_u(\varphi \to \psi)$.

Assume that $V(\varphi \to \psi) \notin P$. We prove that $P \notin V_u(\varphi \to \psi)$. Let us consider the set $P * \{V(\varphi)\}$ and we prove that the set

$$\Gamma = (P * \{V(\varphi)\}) \cup \{V(\neg\psi)\}$$

has the finite intersection property (fip). Suppose the contrary, i.e., $\emptyset \in \Gamma$. Then there exists $U \in P$ such that $(U * V(\varphi)) \cap V(\neg\psi) = \emptyset$. So $U * V(\varphi) \subseteq V(\psi)$, and this implies that $U \subseteq V(\varphi) \Rightarrow V(\psi) = V(\varphi \to \psi)$. It follows $V(\varphi \to \psi) \in P$, which is a contradiction. Then Γ has the fip. Then, by Ultrafilter theorem, there exist an ultrafilter D on X such that $\Gamma \subseteq D$. By Theorem 2, it is follows that there exists Q on X such that $P * Q \subseteq D$, $V(\varphi) \in Q$ and $V(\neg\psi) \in D$. By inductive hypothesis, $Q \in V_u(\varphi)$ and $D \notin V_u(\psi)$. Thus, $P \notin V_u(\varphi \to \psi)$.

2. It follows immediately from 1. ∎

Theorem 19 *Let $\mathcal{M}$ be a model. Then $U_e(\mathcal{M}) = \langle U_e(\mathcal{F}), V_u\rangle$ is a m-saturated model.*

Proof. We prove M1. Let us consider $\Gamma, \Delta \subseteq Fm$ and $P \in Ul(\mathcal{P}(X))$ such tha

$$T_u^{-1}(P) \cap (V_u(\Gamma_0) \times V_u(\Delta_0)) \neq \emptyset$$

for any $\Gamma_0 \in \mathcal{P}_f(\Gamma)$ and for any $\Delta_0 \in \mathcal{P}_f(\Delta)$. Let us consider the sets $\Gamma' = \{V(\varphi) : \varphi \in \Gamma\}$, $\Delta' = \{V(\psi) : \psi \in \Delta\}$, and the filters $F(\Gamma')$ and $F(\Delta')$ in $\mathcal{P}(X)$ generated by Γ' and Δ', respectively. We prove that

$$F(\Gamma') * F(\Delta') \subseteq P. \tag{1}$$

Let $U \in F(\Gamma')$ and $V \in F(\Delta')$. Then there exists $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ and $\{\psi_1, \ldots, \psi_k\} \subseteq \Delta$ such that

$$U * V \subseteq V(\varphi_1 \wedge \ldots \wedge \varphi_n) * V(\psi_1 \wedge \ldots \wedge \psi_k) = V((\varphi_1 \wedge \ldots \wedge \varphi_n) \circ (\psi_1 \wedge \ldots \wedge \psi_k)).$$

By assumption $T_u^{-1}(P) \cap (V_u(\varphi_1 \wedge \ldots \wedge \varphi_n), V_u(\psi_1 \wedge \ldots \wedge \psi_k)) \neq \emptyset$. So, there exist $Q', D' \in Ul(\mathcal{P}(X))$ such that $Q' * D' \subseteq P$, $Q' \in V_u(\varphi_1 \wedge \ldots \wedge \varphi_n)$ and $D' \in V_u(\psi_1 \wedge \ldots \wedge \psi_k$. It follows, $V(\varphi_1 \wedge \ldots \wedge \varphi_n) \in Q'$ and $V(\psi_1 \wedge \ldots \wedge \psi_k) \in D'$. This implies that $V((\varphi_1 \wedge \ldots \wedge \varphi_n) \circ (\psi_1 \wedge \ldots \wedge \psi_k) \in P$. Therefore, (1) is valid. By Theorem 2, there exist $Q, D \in Ul(\mathcal{P}(X))$ such that $Q * D \subseteq P$, $\Gamma' \subseteq Q$ and $\Delta' \subseteq D$. Thus, $T_u^{-1}(P) \cap (V_u(\Gamma) \times V_u(\Gamma)) \neq \emptyset$.

We prove M2. Suppose now that

$$T_u(P) \cap (V_u(\Gamma_0), V_u(\neg\Delta_0)) \neq \emptyset$$

for any $\Gamma_0 \in \mathcal{P}_f(\Gamma)$ and for any $\Delta_0 \in \mathcal{P}_f(\Delta)$. Let us consider the sets $\Gamma' = \{V(\varphi) : \varphi \in \Gamma\}$, $\neg\Delta' = \{V(\neg\psi) : \psi \in \Delta\}$. We prove that

$$(P * \Gamma') \cup \neg\Delta' \tag{2}$$

has the fip. Suppose the contrary. Then there exist $U \in P$, $V(\varphi_1), ..., V(\varphi_n) \in \Gamma'$ and $V(\neg\psi_1), ..., V(\neg\psi_k) \in \neg\Delta'$ such that

$$(U * V(\varphi_1) \cap ... \cap V(\varphi_n)) \cap V(\neg\psi_1) \cap ... \cap V(\neg\psi_k) = \emptyset.$$

This implies that

$$\begin{aligned} U &\subseteq (V(\varphi_1) \cap ... \cap V(\varphi_n)) \Rightarrow V(\psi_1 \vee ... \vee \psi_k) \\ &= V(\varphi_1 \wedge ... \wedge \varphi_n) \Rightarrow V(\psi_1 \vee ... \vee \psi_k) \\ &= V((\varphi_1 \wedge ... \wedge \varphi_n) \rightarrow (\psi_1 \vee ... \vee \psi_k)) \in P. \end{aligned}$$

But by assumption, there exist $Q, D \in Ul(\mathcal{P}(X))$ such that $P * Q \subseteq D$, $V(\varphi_1 \wedge .... \wedge \varphi_n) \in Q$ and $V(\neg(\psi_1 \vee ... \vee \psi_k)) \in D$. Thus, $V((\varphi_1 \wedge ... \wedge \varphi_n) \rightarrow (\psi_1 \vee ... \vee \psi_k)) \notin P$, which is a contradiction. So, (2) is valid. Then by the Ultrafilter theorem and Theorem 2, there exist $Q, D \in Ul(\mathcal{P}(X))$ such that $P * Q \subseteq D$, $\Gamma' \subseteq Q$ and $\neg\Delta' \subseteq D$. Thus, $T_u(P) \cap (V_u(\Gamma) \times V_u(\neg\Delta)) \neq \emptyset$. ■

As a consequence of the above result we have the mentioned characterization of the equivalence between models by means of ultrafilter extensions.

Theorem 20 *Let $\mathcal{M}$ and $\mathcal{M}'$ be two models. Let $a \in \mathcal{M}$ and $b \in \mathcal{M}'$. Then*

$$\mathcal{M}, a \approx \mathcal{M}', b \text{ if and only if } U_e(\mathcal{M}), H(a) \underline{\leftrightarrow} U_e(\mathcal{M}'), H(b).$$

Proof. The proof follows by the following observation.

Let $a \in \mathcal{M}$ and $b \in \mathcal{M}'$. It follows by the assertion 2 of Theorem 18 that

$$\mathcal{M}, a \approx \mathcal{M}', b \text{ if and only if } U_e(\mathcal{M}), H(a) \approx U_e(\mathcal{M}'), H(b),$$

and since both $U_e(\mathcal{M})$ and $U_e(\mathcal{M}')$ are m-saturated by Theorem 19, it follows from Theorem 14 that

$$U_e(\mathcal{M}), H(a) \approx U_e(\mathcal{M}'), H(b) \text{ if and only if } U_e(\mathcal{M}), H(a) \underline{\leftrightarrow} U_e(\mathcal{M}'), H(b)..$$

From these remarks we have the proof of the Theorem. ■

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