ACTAS DEL VI CONGRESO
DR. ANTONIO A. R. MONTEIRO
(2001), Páginas 19–32

The Lagrange-d'Alembert-Poincaré Equations for the Symmetric Rolling Sphere

Hernán Cendra
Departamento de Matemática
Universidad Nacional del Sur, Av. Alem 1254
8000 Bahia Blanca and CONICET, Argentina
uscendra@criba.edu.ar

Ernesto A. Lacomba
Departamento de Matemáticas
UAM - Iztapalapa
Apdo. Postal 55-534, 09340 Mexico, DF, Mexico
lace@xanum.uam.mx

Walter Reartes
Departamento de Matemática
Universidad Nacional del Sur, Av. Alem 1254
8000 Bahia Blanca, Argentina
reartes@mail.uns.edu.ar

Abstract

Nonholonomic systems are described by the Lagrange-d'Alembert principle. The presence of symmetry leads, upon the choice of an arbitrary principal connection, to a reduced variational principle and to the Lagrange-d'Alembert-Poincaré reduced equations. The case of rolling constraints has a long history and it has been the purpose of many works in recent times, in part because of its applications to robotics. In this paper we study the case of a symmetric sphere, that is, a sphere where two of its three moments of inertia are equal, rolling on a plane, using an abelian group of symmetry. The presence of some impulsive constraints and its effect on the reduced variables is also briefly studied.

1 Introduction

Reduction theory for mechanical systems with symmetry has its roots in the classical works on mechanics of Euler, Jacobi, Lagrange, Hamilton, Routh, Poincaré and others. The modern vision of mechanics includes, besides the traditional mechanics of particles and rigid bodies, field theories such as electromagnetism, fluid mechanics, plasma physics, solid mechanics and others of fundamental importance.

A thorough understanding of the several types of symmetries appearing in those theories, sometimes in an obvious way and sometimes in a very subtle way, provides a deep insight, in part because reduction theory can give conservation laws and a description in tems of fewer variables. Much research effort has gone into the development of the symplectic and Poisson view of reduction theory, but recently the Lagrangian view, with an emphasis on the reduction of variational principles has attracted considerable attention, Marsden and Scheurle [1993], Cendra, Marsden and Ratiu (CMR)[2001a].

The universal formalism created by Euler and Lagrange is not applicable to surprisingly simple systems, like those having rolling constraints. The development of a systematic theory generalizing those examples leads to a branch of differential geometry, the geometry of nonholonomic manifolds. However, the systematic treatment of nonholonomic systems and the reduction of d'Alembert principle in the presence of symmetry is relatively recent, Bloch, Krishnaprasad, Marsden, and Murray (BKMM)[1996]. The task of providing an intrinsic geometric formulation of the reduction theory for nonholonomic systems from the point of view of Lagrangian reduction has been established in Cendra, Marsden and Ratiu (CMR)[2001b]. In this work, the reduced Lagrange d'Alembert equations, and in particular, its vertical part, the momentum equation, is written intrinsically using covariant derivatives. The resulting equations are called the *Lagrange-d'Alembert-Poincaré equations.*

We have mentioned only a few references directly related to the present work. However one should be aware that nonholonomic systems have been approached by several people in recent years, using different techniques some of which are given at the end. Among them, we mention for instance, Bloch and Crouch[1992, 1994], Cantrijn, Cortés, de León and de Diego[2000], Cantrijn, de León, Marrero and de Diego [1998], Ibort, de León, Lacomba, de Diego and Pitanga [1977], Koiller[1992], Koon and Marsden[1997b], Koon and Marsden[1997c], Lacomba and Tulczyjew[1990], Neĭmark and Fufaev[1972], Ibort, de León, Lacomba, Marrero, de Diego and Pitanga [2001].

The present short paper is organized as follows. In section 2, we recall the basic facts about Lagrange-d'Alembert-Poincaré equations. In section 3, the main section, we describe the example of the symmetric rolling sphere. It is interesting that this example has an abelian group of symmetry, which leaves invariant both, the Lagrangian and the constraint, namely, the group $SO(2)\times\mathbb{R}^2$. In section 4 we study some impulsive constraints for the case of the symmetric rolling sphere and show their effect at the reduced level.

2 The Lagrange-d'Alembert-Poincaré Equations

The Nonholonomic Connection. Let $\pi\colon Q \to Q/G$ be a principal bundle with structure group G. It is known and easy to prove that there is always a G-invariant metric on Q. In many important physical examples there is a natural way of choosing an invariant metric, representing, for instance, the inertia tensor of the system, see Bloch, Krishnaprasad, Marsden, and Murray [1996]. Now assume that $\mathcal{D}$ is a given invariant distribution on Q. In physical examples this distribution often represents a nonholonomic constraint. We are going to assume the following *dimension assumption*, see CMR[2001b],

$$TQ = \mathcal{D} + \mathcal{V},$$

where $\mathcal{V}$ is the vertical distribution. Let $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$. We can then define the principal connection form $\mathcal{A} : TQ \to \mathfrak{g}$ such that the horizontal distribution $\mathrm{Hor}^{\mathcal{A}} TQ$ satisfies the condition that, for each q, the space $\mathrm{Hor}^{\mathcal{A}} T_qQ$ coincides with the orthogonal complement $\mathcal{H}_q$ of the space $\mathcal{S}_q$ in $\mathcal{D}_q$. This connection is called the *nonholonomic connection*. For each $q \in Q$, let us denote $\mathcal{U}_q$ the orthogonal complement of $\mathcal{S}_q$ in $\mathcal{V}_q$. Then it is easy to see that $\mathcal{U}$ is a smooth distribution and we have the Whitney sum decomposition

$$TQ = \mathcal{H} \oplus \mathcal{S} \oplus \mathcal{U}.$$

We obviously have

$$\mathcal{D} = \mathcal{H} \oplus \mathcal{S}$$

and

$$\mathcal{V} = \mathcal{S} \oplus \mathcal{U}.$$

Under the invariance assumption, all three distributions $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{U}$ are G-invariant, so we can write,

$$TQ/G = \mathcal{H}/G \oplus \mathcal{S}/G \oplus \mathcal{U}/G.$$

The Geometry of the Reduced Bundles. Recall from CMR[2001a] that there is a vector bundle isomorphism

$$\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

where $\tilde{\mathfrak{g}}$ is the adjoint bundle of the principal bundle Q, defined as follows

$$\alpha_{\mathcal{A}}\left([q, \dot{q}]_G\right) = T\pi(q, \dot{q}) \oplus [q, \mathcal{A}(q, \dot{q})]_G,$$

where $(q, \dot{q}) \in TQ$ and the index G denotes equivalence classes under the action of G. Notice that the bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$ does not depend on the connection $\mathcal{A}$, however, the vector bundle isomorphism $\alpha_{\mathcal{A}}$ does depend on $\mathcal{A}$. It is easy to see that

$$\alpha_{\mathcal{A}}(\mathcal{H}/G) = T(Q/G),$$

and

$$\alpha_{\mathcal{A}}(\mathcal{V}/G) = \tilde{\mathfrak{g}}.$$

Define the subbundles $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{u}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{s}} = \alpha_{\mathcal{A}}(\mathcal{S}/G)$$

and

$$\tilde{\mathfrak{u}} = \alpha_{\mathcal{A}}(\mathcal{U}/G)$$

respectively. Clearly, we have,

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{s}} \oplus \tilde{\mathfrak{u}}.$$

Also recall from CMR[2001a,b] that the bundle $\tilde{\mathfrak{g}}$ has a structure defined as follows

a. A Lie algebra structure on each fiber of $\tilde{\mathfrak{g}}$ defined by

$$[[q, \xi_1]_G, [q, \xi_2]_G] = [q, [\xi_1, \xi_2]]_G$$

b. A covariant derivative of curves on $\tilde{\mathfrak{g}}$, given by

$$\frac{D}{Dt}[q(t),\xi(t)]_G = \left[q(t),[-\mathcal{A}(q(t),\dot{q}(t)),\xi(t)]+\dot{\xi}(t)\right]_G.$$

The corresponding connection in $\tilde{\mathfrak{g}}$ is denoted $\tilde{\nabla}^{\mathcal{A}}$.

c. A $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{\mathcal{B}}$ on the base Q/G defined by

$$\tilde{\mathcal{B}}\left((x,\dot{x}_1),(x,\dot{x}_2)\right) = \left[q,\mathcal{B}\left((x,\dot{x}_1)_q^h,(x,\dot{x}_2)_q^h\right)\right]_G$$

where q satisfies $[q]_G = x$, $(x,\dot{x}_i)_q^h$ is the horizontal lift of $(x,\dot{x}_i)$ at the point q, for $i=1,2$, and $\mathcal{B}$ is the curvature of $\mathcal{A}$.

Now let $L\colon TQ \to \mathbb{R}$ be a given invariant Lagrangian. Then it naturally induces a reduced Lagrangian $\ell\colon TQ/G \to \mathbb{R}$. Via the identification given by the vector bundle isomorhism $\alpha_{\mathcal{A}}$ we will often think of ℓ as being a map $\ell\colon T(Q/G)\oplus\tilde{\mathfrak{g}} \to \mathbb{R}$, or, with the usual notation in terms of variables, $\ell(x,\dot{x},\bar{v})$. We should remark that x, $\dot{x}$ and $\bar{v}$ are not independent variables, unless $T(Q/G)$ and $\tilde{\mathfrak{g}}$ are trivial bundles, see CMR[2001a]. Finally, given any connection ∇ on Q/G we have a naturally defined connection $\nabla\oplus\tilde{\nabla}^{\mathcal{A}}$ on $T(Q/G)\oplus\tilde{\mathfrak{g}}$. It is with respect to this connection that the covariant derivatives appearing in the following theorem, like

$$\frac{\partial^C \ell}{\partial x}(x,\dot{x},\bar{v}) \quad\text{and}\quad \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}}(x,\dot{x},\bar{v})$$

should be understood. See for instance Kobayashi and Nomizu[1963] or CMR[2001a, b] for details.

The following theorem CMR[2001b] is the main result that we need to study the symmetric rolling sphere in the next section.

Theorem 2.1 *Let $q(t)$ be a curve in Q such that $(q(t),\dot{q}(t)) \in \mathcal{D}_{q(t)}$ for all t and let $(x(t),\dot{x}(t),\bar{v}(t)) = \alpha_{\mathcal{A}}([q(t),\dot{q}(t)]_G)$ be the corresponding curve in $T(Q/G)\oplus\tilde{\mathfrak{s}}$. The following conditions are equivalent.*

(i) *The* ***Lagrange-d'Alembert principle*** *holds:*

$$\delta\int_{t_0}^{t_1} L(q,\dot{q})dt = 0$$

for variations δq of the curve q such that $\delta q(t_i)=0$, for $i=0,1$, and $\delta q(t)\in\mathcal{D}_{q(t)}$, for all t.

(ii) *The* ***reduced Lagrange-d'Alembert principle*** *holds: The curve $x(t)\oplus\bar{v}(t)$ satisfies*

$$\delta\int_{t_0}^{t_1} \ell\left(x(t),\dot{x}(t),\bar{v}(t)\right)dt = 0,$$

for variations $\delta x\oplus\delta^{\mathcal{A}}\bar{v}$ of the curve $x(t)\oplus\bar{v}(t)$, where $\delta^{\mathcal{A}}\bar{v}$ has the form

$$\delta^{\mathcal{A}}\bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v},\bar{\eta}] + \tilde{\mathcal{B}}(\delta x,\dot{x}),$$

with the boundary conditions $\delta x(t_i)=0$ and $\bar{\eta}(t_i)=0$, for $i=0,1$, and where $\bar{\eta}(t)\in\tilde{\mathfrak{s}}_{x(t)}$.

(iii) *The following* ***vertical Lagrange-d'Alembert-Poincaré equations****, corresponding to vertical variations, hold:*

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v})\Big|_{\tilde{\mathfrak{s}}} = \mathrm{ad}^*_{\bar{v}}\frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v})\Big|_{\tilde{\mathfrak{s}}},$$

and the ***horizontal Lagrange-d'Alembert-Poincaré equations****, corresponding to horizontal variations,*

$$\frac{\partial^C \ell}{\partial x}(x,\dot{x},\bar{v}) - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}}(x,\dot{x},\bar{v}) = \left\langle \frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v}), \mathbf{i}_{\dot{x}}\tilde{\mathcal{B}}(x)\right\rangle.$$

hold.

In part (ii) of this theorem, if $\bar{v} = [q,v]_G$ with $v = \mathcal{A}(q,\dot{q})$ then $\bar{\eta}$ can be always written $\bar{\eta} = [q,\eta]_G$, and the condition $\bar{\eta}(t_i) = 0$ for $i = 0,1$, is equivalent to the condition $\eta(t_i) = 0$ for $i = 0,1$. Also, if $x(t) = [q(t)]_G$ and $\bar{v} = [q,v]_G$ where $v = \mathcal{A}(q,\dot{q})$, then variations $\delta x \oplus \delta^{\mathcal{A}}\bar{v}$ such that

$$\delta^{\mathcal{A}}\bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v},\bar{\eta}] \equiv \frac{D[q,\eta]_G}{Dt} + [q,[v,\eta]]_G$$

with $\bar{\eta}(t_i) = 0$ (or, equivalently, $\eta(t_i) = 0$) for $i = 0,1$, and $\bar{\eta}(t) \in \tilde{\mathfrak{s}}_{x(t)}$, correspond exactly to vertical variations δq of the curve q such that $\delta q(t_i) = 0$ for $i = 0,1$, and $\delta q(t) \in \mathcal{S}_{q(t)}$, while variations $\delta x \oplus \delta^{\mathcal{A}}\bar{v}$ such that

$$\delta^{\mathcal{A}}\bar{v} = \tilde{\mathcal{B}}(\delta x, \dot{x})$$

with $\delta x(t_i) = 0$ for $i = 0,1$, correspond exactly to horizontal variations δq of the curve q such that $\delta q(t_i) = 0$.

3 The Symmetric Rolling Sphere

Kinematics of the Rigid Body. The configuration space for the rigid body is the group $SO(3)$. The motion of the rigid body is given by a curve $A(t)$ on $SO(3)$. The *space angular velocity* $\hat{\omega}$ and the *body angular velocity* $\hat{\Omega}$ are elements of the Lie algebra $\mathfrak{so}(3)$ and they are defined by the conditions $\dot{A} = A\hat{\Omega} = \hat{\omega}A$.

We recall that there is a natural identification $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\hat{x} = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix},$$

where $x = x^1\,\mathbf{e}_1 + x^2\,\mathbf{e}_2 + x^3\,\mathbf{e}_3$.

We have the formulas

$$\widehat{x \times y} = [\hat{x},\hat{y}], \quad x\cdot y = -\frac{1}{2}\operatorname{tr}\hat{x}\hat{y} \quad \text{and} \quad \hat{x}\,y = x \times y.$$

Besides, if A is any element of $SO(3)$ and x is any element of $\mathbb{R}^3$ we have

$$\widehat{Ax} = A\hat{x}A^{-1}.$$

For any motion $A(t)$, define $z(t) = A(t)\,\mathbf{e}_3$. Then

$$\dot{z} = \dot{A}\,\mathbf{e}_3 = \hat{\omega} z = \omega \times z.$$

We have that $\langle \omega, z\rangle = \langle \Omega, \mathbf{e}_3\rangle = \Omega^3$, and that $A(\Omega^1 \hat{\mathbf{e}}_1 + \Omega^2 \hat{\mathbf{e}}_2)A^{-1} = \widehat{(z \times \dot{z})}$. Therefore the space velocity ω can be written $\omega = A\Omega$ and then $\omega = \Omega^3 z + z \times \dot{z}$. This gives a decomposition of ω as a sum of its component parallel to z plus its component normal to z.

Kinematics of the Rolling Sphere. The configuration space for the rolling sphere is

$$SO(3) \times \mathbb{R}^2.$$

We shall often identify the factor $\mathbb{R}^2$ with the subspace of $\mathbb{R}^3$ defined by $x^3 = 0$. The nonholonomic constraint is given by the distribution

$$\mathcal{D}_{(A,x)} = \{(A, x, \dot{A}, \dot{x}) | \hat{\omega} r \mathbf{e}_3 - \dot{x} = 0\}$$

The symmetry group is $SO(2) \times \mathbb{R}^2$, where $SO(2)$ is identified with the set of elements of $SO(3)$ of the type

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $SO(2) \times \mathbb{R}^2$ acts on $SO(3) \times \mathbb{R}^2$ by

$$(A, x)(g, a) = (Ag, x + a).$$

With this action $SO(3) \times \mathbb{R}^2$ becomes a right $SO(2) \times \mathbb{R}^2$–principal bundle.

The map $\pi \colon SO(3) \times \mathbb{R}^2 \to S^2$ given by $\pi(A, x) = A\,\mathbf{e}_3$ is a submersion. We have an identification $SO(3) \times \mathbb{R}^2 / SO(2) \times \mathbb{R}^2 \equiv S^2$ given by

$$[A, x]_{SO(2)\times\mathbb{R}^2} \equiv A\mathbf{e}_3$$

The vertical distribution $\mathcal{V}$ is given by

$$\mathcal{V}_{(A,x)} = \left\{(A, x, \dot{A}, \dot{x}) | \dot{A} = A\xi\hat{\mathbf{e}}_3, \xi \in \mathbb{R}\right\}.$$

The vector bundle $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$ is given by

$$\mathcal{S}_{(A,x)} = \{(A, x, A\xi\hat{\mathbf{e}}_3, \xi z \times r\mathbf{e}_3) | \xi \in \mathbb{R}\}.$$

The adjoint bundle $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is a trivial bundle and we have an identification

$$\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2} \equiv S^2 \times (\mathfrak{so}(2) \times \mathbb{R}^2)$$

given by

$$[A, x, \xi\hat{\mathbf{e}}_3, a]_{SO(2)\times\mathbb{R}^2} \equiv (z, \xi\hat{\mathbf{e}}_3, a),$$

where $z = A\,\mathbf{e}_3$.

Now we define the connection $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A}(A,x,\dot{A},\dot{x}) &= \left(\Omega^3\hat{\mathrm{e}}_3, \dot{x} - A(\Omega^1\hat{\mathrm{e}}_1 + \Omega^2\hat{\mathrm{e}}_2)A^{-1}r\mathrm{e}_3\right)\\ &= \left(\Omega^3\hat{\mathrm{e}}_3, \dot{x} - (z\times\dot{z})\times r\mathrm{e}_3\right).\end{aligned}$$

We can easily see that the horizontal distribution $\mathcal{H}$ for this connection satisfies $\mathcal{D} = \mathcal{S}\oplus\mathcal{H}$.

The vector bundle isomorphism $\alpha_{\mathcal{A}}$ described in CMR[2001a,b] is given by

$$\alpha_{\mathcal{A}}\left([A,x,\dot{A},\dot{x}]_{SO(2)\times\mathbb{R}^2}\right) = (z,\dot{z})\oplus(z,\bar{v}),$$

where $\bar{v} = (v_0\hat{\mathrm{e}}_3, v_1)$, with $v_0 = \Omega^3$ and $v_1 = \dot{x} - (z\times\dot{z})\times r\mathrm{e}_3$.

The subbundle $\tilde{\mathfrak{s}} \subset \widetilde{\mathfrak{so}(2)\times\mathbb{R}^2}$ is given by

$$\tilde{\mathfrak{s}} = \left\{(z,\xi\hat{\mathrm{e}}_3,\xi z\times r\mathrm{e}_3) | \xi\in\mathbb{R}, z\in S^2\right\}$$

Now we shall describe the structure of the bundle $\widetilde{\mathfrak{so}(2)\times\mathbb{R}^2}$. First of all, the Lie algebra structure on each fiber of $\widetilde{\mathfrak{so}(2)\times\mathbb{R}^2}$ is abelian because the Lie algebra $\mathfrak{so}(2)\times\mathbb{R}^2$ is abelian.

Let $(z,\xi\hat{\mathrm{e}}_3,a)$ be a curve on $\widetilde{\mathfrak{so}(2)\times\mathbb{R}^2}$. Using the formula for the covariant derivative, and using the fact that the group $SO(2)\times\mathbb{R}^2$ is abelian, we see that the covariant derivative of this curve is given by

$$\frac{D(z,\xi\hat{\mathrm{e}}_3,a)}{Dt} = (z,\dot{\xi}\hat{\mathrm{e}}_3,\dot{a}).$$

Now we shall calculate $\tilde{\mathcal{B}}$. To calculate the curvature $\mathcal{B}$, let

$$X_i(A,x) = \left(A, x, A\hat{\Omega}_i, A(\Omega_i^1\hat{\mathrm{e}}_1 + \Omega_i^2\hat{\mathrm{e}}_2)A^{-1}r\mathrm{e}_3\right),$$

where $\hat{\Omega}_i = \Omega_i^1\hat{\mathrm{e}}_1 + \Omega_i^2\hat{\mathrm{e}}_2$, $i = 1,2$ are fixed, be given horizontal vector fields. Consider the decomposition $X_i(A,x) = X_{i1}(A,x) + X_{i2}(A,x)$, $i=1,2$ where $X_{i1}(A,x) = (A,x,A\hat{\Omega}_i,0)$ and $X_{i2}(A,x) = (A,x,0,A(\Omega_i^1\hat{\mathrm{e}}_1 + \Omega_i^2\hat{\mathrm{e}}_2)A^{-1}r\hat{\mathrm{e}}_3)$. It is easy to see that the Lie bracket $[X_1,X_2]$ is given by the formula

$$[X_1,X_2] = (DX_{21}\cdot X_{11} - DX_{11}\cdot X_{21}, DX_{22}\cdot X_{11} - DX_{12}\cdot X_{21}).$$

After some calculations we obtain

$$[X_1,X_2] = \left(A\left[\hat{\Omega}_1,\hat{\Omega}_2\right], 2A\left[\hat{\Omega}_1,\hat{\Omega}_2\right]A^{-1}r\mathrm{e}_3.\right)$$

Note that $\left[\hat{\Omega}_1,\hat{\Omega}_2\right] = \left[\hat{\Omega}_1,\hat{\Omega}_2\right]^3\hat{\mathrm{e}}_3$. Since the group is abelian the curvature is given by

$$\mathcal{B} = -\mathcal{A}\left([X_1,X_2]\right).$$

See Kobayashi and Nomizu[1963] for details. Therefore, we obtain

$$\mathcal{B} = -\left((\Omega_1\times\Omega_2)^3\hat{\mathrm{e}}_3, 2A\left[\hat{\Omega}_1,\hat{\Omega}_2\right]A^{-1}r\mathrm{e}_3\right).$$

Let $\dot{z}_i = A\hat{\Omega}_i \mathbf{e}_3 = \omega_i \times z$ and note that $\omega_i = z \times \dot{z}_i$ and that $\langle \omega_i, z \rangle = 0$, for $i = 1, 2$. Then we have

$$\begin{aligned}(\Omega_1 \times \Omega_2)^3 &= \langle \Omega_1 \times \Omega_2, \mathbf{e}_3 \rangle \\ &= \langle \omega_1 \times \omega_2, z \rangle \\ &= \langle (z \times \dot{z}_1) \times (z \times \dot{z}_2), z \rangle \\ &= \langle \dot{z}_1 \times \dot{z}_2, z \rangle\end{aligned}$$

and also

$$\begin{aligned}2A\left(\widehat{\Omega_1 \times \Omega_2}\right) A^{-1} r\mathbf{e}_3 &= 2(\omega_1 \times \omega_2) \times r\mathbf{e}_3 \\ &= 2\left((z \times \dot{z}_1) \times (z \times \dot{z}_2)\right) \times r\mathbf{e}_3 \\ &= 2(\dot{z}_1 \times \dot{z}_2) \times r\mathbf{e}_3.\end{aligned}$$

From this we easily deduce that

$$\tilde{\mathcal{B}}(\dot{z}_1, \dot{z}_2) = -\left(z, \langle \dot{z}_1 \times \dot{z}_2, z \rangle \hat{\mathbf{e}}_3, 2(\dot{z}_1 \times \dot{z}_2) \times r\mathbf{e}_3\right)$$

Dynamics of the Rolling Sphere. The Lagrangian for a general nonhomogeneous rolling sphere is given by

$$L(A, x, \dot{A}, \dot{x}) = \frac{1}{2}\langle \mathbf{I}\Omega, \Omega \rangle + \frac{1}{2}M\dot{x}^2,$$

where

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

is the inertia tensor and M is the mass of the sphere.

In this article we will consider only the case of an axially symmetric sphere where $I_1 = I_2$. This has the advantage that the Lagrangian is invariant under the right action of the abelian group $SO(2) \times \mathbb{R}^2$, which also leaves the constraint $\mathcal{D}$ invariant.

The reduced Lagrangian $\ell(z, \dot{z}, \bar{v})$ is given by

$$\ell(z, \dot{z}, \bar{v}) = \frac{1}{2}I_1\dot{z}^2 + \frac{1}{2}I_3 v_0^2 + \frac{1}{2}M\left(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3\right)^2$$

where $\bar{v} = (v_0\hat{\mathbf{e}}_3, v_1)$, with $v_1 = v_1^1\mathbf{e}_1 + v_1^2\mathbf{e}_2$.

Now we introduce the following dimensionless quantities

$$\alpha = \frac{I_3}{I_1}, \quad \beta = \frac{Mr^2}{I_1} \quad \text{and} \quad y = \frac{x}{r}.$$

Since the group is abelian the vertical Lagrange–d'Alembert–Poincaré equation becomes

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}}\bigg|_{\bar{s}} = 0.$$

We shall often identify $\xi\hat{\mathbf{e}}_3 \equiv \xi$; thus an element $(z, v_0\hat{\mathbf{e}}_3, v_1)$ of $\tilde{\mathfrak{s}}$ is written simply (z, v_0, v_1). We have

$$\frac{\partial \ell}{\partial \bar{v}} = \left(\frac{\partial \ell}{\partial v_0}, \frac{\partial \ell}{\partial v_1}\right) \tag{1}$$

$$= (z, I_3 v_0, M(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3)) \tag{2}$$

Since $\bar{v} \in \tilde{\mathfrak{s}}$ we have $\bar{v} = (z, v_0, v_0 z \times r\mathbf{e}_3)$. Since $\tilde{\mathfrak{s}}$ is 1 dimensional we have that $(z, 1, z \times r\mathbf{e}_3)$ is a nonvanishing section of $\tilde{\mathfrak{s}}$ that generates $\tilde{\mathfrak{s}}$. Then the vertical Lagrange–d'Alembert–Poincaré equation becomes

$$\frac{D}{Dt}\left(z, I_3 v_0, M\left(v_0 z \times r\mathbf{e}_3 + (z \times \dot{z}) \times r\mathbf{e}_3\right)\right) \cdot (z, 1, z \times r\mathbf{e}_3) = 0.$$

Since $\omega = v_0 z + z \times \dot{z}$ the vertical equation becomes

$$I_3 \dot{v}_0 + M(\dot{\omega} \times r\mathbf{e}_3) \cdot (z \times r\mathbf{e}_3) = 0.$$

Since $\ddot{x} = \dot{\omega} \times r\mathbf{e}_3$ we obtain

$$I_3 \dot{v}_0 + M\ddot{x} \cdot (z \times r\mathbf{e}_3) = 0$$

or

$$I_3 \dot{v}_0 = \langle z, M\ddot{x} \times r\mathbf{e}_3 \rangle.$$

Observe that $M\ddot{x} \times r\mathbf{e}_3$ is the momentum of the force $M\ddot{x}$ at the point of contact with respect to the center of the sphere, thus $\langle z, M\ddot{x} \times r\mathbf{e}_3 \rangle$ is the momentum of that force with respect to the axis z.

In terms of the adimensional quantities introduced above, the vertical equations become

$$\alpha \dot{v}_0 = \beta \langle \ddot{y} \times \mathbf{e}_3, z \rangle,$$

where $\dot{y} = \omega \times \mathbf{e}_3$, $\omega = v_0 z + z \times \dot{z}$, and therefore, $\ddot{y} = (\dot{v}_0 z + v_0 \dot{z} + z \times \ddot{z}) \times \mathbf{e}_3$.

Now let us calculate the horizontal Lagrange–d'Alembert–Poincaré equation. We can decompose the reduced Lagrangian as follows $\ell(z, \dot{z}, \bar{v}) = \ell_h(z, \dot{z}, \bar{v}) + \ell_v(z, \dot{z}, \bar{v})$, where

$$\ell_h(z, \dot{z}, \bar{v}) = \frac{1}{2} I_1 \dot{z}^2$$

and

$$\ell_v(z, \dot{z}, \bar{v}) = \frac{1}{2} I_3 v_0^2 + \frac{1}{2} M \left(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3\right)^2.$$

A standard calculation shows that

$$\frac{\partial \ell_h}{\partial z} - \frac{D}{Dt}\frac{\partial \ell_h}{\partial \dot{z}} = -I_1 \nabla_{\dot{z}} \dot{z},$$

where we have used the identification of tangent vectors to S^2 with covectors via the standard metric of S^2.

We have

$$\frac{\partial \ell_v}{\partial z} \cdot \delta z = M\left(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3\right) \cdot \left((\delta z \times \dot{z}) \times r\mathbf{e}_3\right).$$

Since $\bar{v} \in \tilde{\mathfrak{s}}_z$, we have $\bar{v} = (z, v_0, v_0 z \times r\mathbf{e}_3)$, therefore we obtain

$$\frac{\partial \ell_v}{\partial z} \cdot \delta z = M\left(\omega \times r\mathbf{e}_3\right) \cdot \left((\delta z \times \dot{z}) \times r\mathbf{e}_3\right).$$

Using the fact that $\dot{x} = \omega \times r\mathbf{e}_3$ we deduce that

$$\frac{\partial \ell_v}{\partial z} \cdot \delta z = \delta z \cdot \left(\dot{z} \times (r\mathbf{e}_3 \times M\dot{x})\right).$$

We have

$$\frac{\partial \ell_v}{\partial \dot{z}} \cdot \delta \dot{z} = M\left(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3\right) \cdot \left((z \times \delta\dot{z}) \times r\mathbf{e}_3\right).$$

Since $\bar{v} \in \tilde{\mathfrak{s}}_z$, we have $\bar{v} = (z, v_0, v_0 z \times r\mathbf{e}_3)$, therefore we obtain

$$\begin{aligned}\frac{\partial \ell_v}{\partial \dot{z}} \cdot \delta \dot{z} &= M\left(\omega \times r\mathbf{e}_3\right) \cdot \left((z \times \delta\dot{z}) \times r\mathbf{e}_3\right) \\ &= \delta\dot{z} \cdot \left((r\mathbf{e}_3 \times M\dot{x}) \times z\right).\end{aligned}$$

The horizontal Lagrange–d'Alembert–Poincaré equation is

$$\frac{\partial \ell}{\partial z} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{z}} = \frac{\partial \ell}{\partial \bar{v}}\left(\tilde{\mathcal{B}}(\dot{z}, \cdot)\right)$$

Collecting the above results the left hand side of this equation is

$$\frac{\partial \ell}{\partial z} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{z}} = -I_1 \nabla_{\dot{z}}\dot{z} + 2Mr^2\left(\dot{z} \times (\mathbf{e}_3 \times (\omega \times \mathbf{e}_3))\right) + z \times (r\mathbf{e}_3 \times M\ddot{x}).$$

and the right hand side is

$$\begin{aligned}&\frac{\partial \ell}{\partial \bar{v}}\left(\tilde{\mathcal{B}}(\dot{z}, \delta z)\right) = \\ &\qquad (z, I_3 v_0, M(v_1 + (z \times \dot{z}) \times r\mathbf{e}_3)) \cdot (z, -\langle \dot{z} \times \delta z, z\rangle, -2(\dot{z} \times \delta z) \times r\mathbf{e}_3))\end{aligned}$$

Since $\bar{v} \in \tilde{\mathfrak{s}}_z$, we have $v_1 = v_0 z \times r\mathbf{e}_3$. After some calculations, we obtain the horizontal equation in the form

$$I_1 \nabla_{\dot{z}}\dot{z} = I_3 v_0 (z \times \dot{z}) + Mr(\ddot{x} \times \mathbf{e}_3) \times z.$$

In terms of the dimensionless quantities defined before we obtain the horizontal Lagrange–d'Alembert–Poincaré equation

$$\nabla_{\dot{z}}\dot{z} = \alpha v_0 (z \times \dot{z}) + \beta(\ddot{y} \times \mathbf{e}_3) \times z.$$

where ∇ is the covariant derivative with respect to the standard metric in S^2. Recall that $\nabla_{\dot{z}}\dot{z} = \ddot{z} - \langle z, \ddot{z}\rangle z$, and therefore $z \times \ddot{z} = z \times \nabla_{\dot{z}}\dot{z}$.

Summarizing, we have obtained the following system of reduced equations in terms of the variables z and v_0

$$\begin{aligned}\alpha\dot{v}_0 &= \beta\langle\ddot{y}\times\mathbf{e}_3, z\rangle\\ \nabla_{\dot{z}}\dot{z} &= \alpha v_0(z\times\dot{z}) + \beta(\ddot{y}\times\mathbf{e}_3)\times z\\ \omega &= v_0 z + z\times\dot{z}\\ \dot{y} &= \omega\times\mathbf{e}_3\end{aligned}$$

The previous reduced system of equations, of which the first two are the vertical and horizontal Lagrange–d'Alembert–Poincaré equations respectively, completely describes the motion in terms of the variables $(z, v_0) \in S^2\times\mathbb{R}$. By transforming this reduced system appropriately and using the fact that the reduced Lagrangian $\ell(z,\dot{z},\bar{v})$ is a preserved quantity, we obtain an equivalent single second order differential equation on S^2. We shall not write a detailed description of this equation in the present paper.

4 Impulsive Constraints

The dynamics of systems with impulsive constraints has been widely treated in classical and also recent books, Painlevé[1930], Appell[1953], Neĭmark and Fufaev[1972]. Geometric aspects of this kind of questions have been studied recently for instance in Lacomba and Tulczyjew[1990], Ibort et al.[1998], Ibort et al.[2001.]

It seems also interesting to have direct information on the effect of impulsive constraints on reduced variables, which we shall show next. In this section we consider a simple example, namely, the elastic collision of the symmetric rolling sphere on the plane, against a vertical rough wall located at $x^1 = r$. In other words, the allowed values x of the point of contact of the sphere with the plane, must satisfy $x^1 \leq 0$, since the radius of the sphere is r. We shall use the dimensionless quantities introduced above, so $r = 1$, $I_1 = 1$, $I_3 = \alpha$ and $M = \beta$. With normalized coordinates y in the plane, the configuration constraint becomes $y^1 \leq 0$.

Let us assume, with no essential loss of generality, that, an instant before the moment the sphere hits the vertical wall, the velocity in configuration space is $(A_0, y_0, \dot{A}_0, \dot{y}_0) = (A_0, 0, \hat{\omega}_0 A_0, \dot{y}_0) \equiv (\omega_0, \dot{y}_0)$, where $\dot{y}_0^1 \geq 0$. We want to determine the velocity an instant after the collision, $(A_1, y_1, \dot{A}_1, \dot{y}_1) = (A_0, 0, \hat{\omega}_1 A_0, \dot{y}_1) \equiv (\omega_1, \dot{y}_1)$.

The intersection of the constraint $\mathcal{D}$ and that given by the rolling of the sphere on the plane $y^1 = 1$, at the point (A_0, y_0), is the 1-dimensional subspace $\mathcal{C}$ of $\mathcal{D}$ generated by the vector $(A_0, 0, \hat{u}A_0, v) \equiv (u, v)$ where $u = (1,0,1)$ and $v = u\times\mathbf{e}_3$, that is, $v = (0,-1)$. We need to find the decomposition of $(\omega_0, \dot{y}_0)$ as the sum of a component parallel to $\mathcal{C}$ plus a component orthogonal to $\mathcal{C}$, with respect to the metric given by the kinetic energy. We know that this kinetic energy metric at any tangent vector $(A_0, 0, \hat{\omega}A_0, \dot{y})$ is given by

$$2L(A_0, 0, \hat{\omega}A_0, \dot{y}) = \dot{z}^2 + \alpha v_0^2 + \beta\dot{y}^2,$$

where $z_0 = A_0\mathbf{e}_3$, $\dot{z} = \omega\times z_0$ and $v_0 = \langle\omega, z_0\rangle$. It is clear that

$$2L(A_0, 0, \hat{\omega}A_0, \dot{y}) = \langle\alpha v_0 z_0 + z_0\times\dot{z}, v_0 z_0 + z_0\times\dot{z}\rangle + \langle\beta\dot{y}, \dot{y}\rangle.$$

To be precise,let us call $K(\cdot,\cdot)$ on $T_{(A_0,0)}(SO(3)\times\mathbb{R})$ the metric associated to the kinetic energy, so that we can write

$$K\left((A_0, 0, \hat{\omega}A_0, \dot{y}), (A_0, 0, \hat{\omega}A_0, \dot{y})\right) = 2L(A_0, 0, \hat{\omega}A_0, \dot{y}).$$

Then we have that the component of $(\omega_0, \dot{y}_0)$ parallel to (u, v) with respect to the metric given by the kinetic energy is given by $\lambda(u, v)$, where

$$\lambda = \frac{K\left((\omega_0, \dot{y}_0), (u, v)\right)}{K\left((u, v), (u, v)\right)}.$$

We can easily see that

$$K\left((\omega_0, \dot{y}_0), (u, v)\right) = \langle \alpha v_{00} z_0 + z_0 \times \dot{z}_0, u\rangle + \langle \beta \dot{y}_0, v\rangle$$

and

$$K\left((u, v), (u, v)\right) = u^2 + (\alpha - 1)\langle u, z_0\rangle^2 + \beta\langle v, v\rangle,$$

where $v_{00} = \langle \omega_0, z_0\rangle$ and $\dot{z}_0 = \omega_0 \times z_0$.

For an elastic collision we have that the component of $(\omega_0, \dot{y}_0)$ parallel to (u, v) is preserved while the component normal to (u, v) changes its sign. Therefore, we obtain

$$(\omega_1, \dot{y}_1) = 2\lambda(u, v) - (\omega_0, \dot{y}_0).$$

From this it follows immediately that $\Delta(\omega, \dot{y}) = (\omega_1, \dot{y}_1) - (\omega_0, \dot{y}_0)$ can be written as follows

$$\Delta(\omega, \dot{y}) = 2\left(\lambda(u, v) - (\omega_0, \dot{y}_0)\right)$$

Now we shall see the effect of the collision on the reduced variables (z, v_0). It is easy to see that z does not change during the collision. On the other hand we have, in general, $v_0 = \langle \omega, z\rangle$. The difference

$$\Delta(\dot{z}, v_0) = (\dot{z}_1, v_{01}) - (\dot{z}_0, v_{00})$$

where $v_{01} = \langle \omega_1, z_0\rangle$, is given by

$$\Delta(\dot{z}, v_0) = (\Delta\omega \times z_0, \langle \Delta\omega, z_0\rangle)$$

Since

$$\Delta\omega = 2\left(\lambda u - \omega_0\right)$$

we obtain,

$$\Delta(\dot{z}, v_0) = 2\left(\lambda(-z_0^3, z_0^1 - z_0^3, z_0^2) - \omega_0 \times z_0, \lambda(z_0^1 + z_0^3) - \langle \omega_0, z_0\rangle\right).$$

Since we know that $\omega_0 = v_{00} z_0 + z_0 \times \dot{z}_0$ and $\dot{y}_0 = \omega_0 \times \mathbf{e}_3$, it follows that the value of the reduced variables $(\dot{z}_1, v_{01})$ after the collision, only depends on the value of the reduced variables $(\dot{z}_0, v_{00})$ before the collision.

The particular case in which $\alpha = 1$, is easier, and has been studied long ago. See for instance Neĭmark and Fufaev [1972] and Ibort et. al [2001]. For instance, in Ibort et. al [2001], the ball is assumed to stay in the half space $x_1 \geq r$, in our notation. Then, to compare with the results of the present paper, we must take $u = (-1, 0, 1)$ and $v = (0, 1)$.

Acknowledgments We thank Jerry Marsden for his invaluable comments and references. We also thank David Schneider for sending us his recent thesis..The work of E. Lacomba was partially supported by CONACYT (Mexico) grant 32167-E.

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