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DISCRETE CONNECTIONS ON PRINCIPAL BUNDLES: ABELIAN GROUP CASE

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Dedicated to our friend Hernán Cendra

ABSTRACT. In this note we consider a few interesting properties of discrete connections on principal bundles when the structure group of the bundle is an abelian Lie group. In particular, we show that the discrete connection form and its curvature can be interpreted as singular 1 and 2 cochains respectively, with the curvature being the coboundary of the connection form. Using this formalism we prove a discrete analogue of a formula for the holonomy around a loop given by Marsden, Montgomery and Ratiu for (continuous) connections in a similar setting.

1. INTRODUCTION

Principal bundles are used to state many important questions in Geometry and in Physics. One versatile tool in the study of those bundles, and in finding answers to the questions, are the principal connections and the "connection package": curvature, parallel transport and holonomy. Curvature and holonomy are closely related notions, as shown, for example, by the Ambrose–Singer Theorem (Thm. 8.1 in [KN96]). When the structure group G of the (left) principal bundle $\pi : Q \to M$ is abelian and $\mathscr{A}$ is a principal connection on π, a well known formula (see, for instance, p. 41 of [MMR90]) gives a direct expression for the holonomy of $\mathscr{A}$ around a loop ρ in M in terms of an integral of $\mathscr{A}$ over ρ or, in case ρ is the boundary of a surface σ, the integral over σ of the curvature of $\mathscr{A}$. This kind of formula is very useful for many applications: for instance, it allows the control of the "displacement" in a fiber by choosing an adequate loop ρ and parallel-transporting along ρ. It is also useful in the reconstruction of the dynamics of some systems on Q that have G as a symmetry group and whose (reduced) dynamics is known in $M := Q/G$ (see §5 of [MMR90]). In this setting the holonomy is also known as the *geometric phase* (see, for instance, [Sim83]).

More precisely, if $V \subset M$ is an open subset and $s : V \to Q$ is a smooth local section of π, let $\rho : [0,1] \to M$ be a continuous loop contained in V and $\overline{m} := \rho(0)$; pick $\overline{q} \in Q|_{\overline{m}}$, the fiber of Q over $\overline{m}$, and define $\Phi_{\mathscr{A}}(\rho,\overline{q}) \in G$ by

$$\Phi_{\mathscr{A}}(\rho,\overline{q}) := \exp_G\left(-\int_\rho \mathscr{A}^s\right), \tag{1.1}$$

where $\mathscr{A}^s := s^*(\mathscr{A})$ is the local expression of $\mathscr{A}$ in the trivialization induced by s (hence a 1-form on V with values in $\mathfrak{g} := Lie(G)$) and $\exp_G : \mathfrak{g} \to G$ is the Lie-theoretic exponential map. If, in addition, ρ is the boundary of a surface σ contained in V, by Stokes' Theorem, we have

$$\Phi_{\mathscr{A}}(\rho,\overline{q}) = \exp_G\left(-\int_\sigma \mathscr{B}^s\right), \tag{1.2}$$

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for $\mathscr{B}^s := s^*(\mathscr{B}) = d\mathscr{A}^s$, where $\mathscr{B}$ is the curvature 2-form of $\mathscr{A}$. These formulas are relevant because $\Phi_{\mathscr{A}}(\rho,\overline{q})$ is the phase gained by $\overline{q}$ on parallel transport around the loop ρ. In other words, if $\mathrm{PT}_{\mathscr{A}}(\rho) : Q|_{\overline{m}} \to Q|_{\overline{m}}$ is the parallel transport over ρ operator (with respect to $\mathscr{A}$), then

$$\mathrm{PT}_{\mathscr{A}}(\rho)(\overline{q}) = l^Q_{\Phi_{\mathscr{A}}(\rho,\overline{q})}(\overline{q}),$$

where l^Q_g is the left G-action on Q defined by the principal G-bundle structure.

As we mentioned above, connections are useful in the study of certain symmetric dynamical systems, for example, the mechanical systems as considered in [AM78]. In many instances, it is essential to consider numerical approximations to those systems, which can be thought of as discrete-time dynamical systems (see [MW01]). In this context it is natural to replace the tangent bundle TQ with $Q \times Q$, its “discrete version”[1]. If, in addition, there is a symmetry group G acting on Q in such a way that the quotient map $\pi : Q \to Q/G$ is a principal G-bundle, in the continuous case, a principal connection $\mathscr{A}$ on π can be used to decompose all elements of TQ into vertical an horizontal parts. Discrete connections were introduced by M. Leok, J. Marsden and A. Weinstein in [LMW05] and [Leo04] and, later, refined by some of us in [FZ13] in order to have a geometric way of splitting elements of $Q \times Q$ into “vertical” and “horizontal” parts in a way that was suitable to perform the symmetry reduction for the discrete time mechanical systems. With this idea in mind, the goal of this paper is to prove formulas analogous to (1.1) and (1.2) for discrete connections on a principal G-bundle, when G is abelian. As for (continuous) connections, a natural application of such formulas is to control the dynamics of symmetric dynamical systems.

Even though there are many conceptual parallels between connections and discrete connections, the fact that (for abelian G) the curvature (2-form) $\mathscr{B}$ is related to the connection 1-form $\mathscr{A}$ by $\mathscr{B} = d\mathscr{A}$ has no equivalent in the discrete world: both the discrete connection form $\mathscr{A}_d$ and its curvature $\mathscr{B}_d$ are G-valued functions. In order to reach our stated goal we introduce a formalism that allows us to view $\mathscr{A}_d$ and $\mathscr{B}_d$ as singular cochains $[\mathscr{A}_d]$ and $[\mathscr{B}_d]$ such that $[\mathscr{B}_d] = \delta[\mathscr{A}_d]$. It is within this framework that we obtain the discrete holonomy phase formulas (4.1) and (4.2). We mention that, in this setting, what we call integration is the natural pairing of singular cochains and chains; this idea is in line with the fact that the integral of differential forms over manifolds is a realization of the pairing between cochains of differential forms —the elements of the de Rham complex— and singular chains on a manifold. There is still a twist in that (1.1) and (1.2) involve integration in $\mathfrak{g}$ while (4.1) and (4.2) use integration of G-valued cochains. Thus, we reformulate our G-valued singular cochains as $\mathfrak{g}$-valued ones and are able to obtain (5.2) and (5.3), the exact analogue of the continuous phase formulas.

The plan for the paper is as follows. In Section 2 we recall the relevant notions of discrete connection on a principal G-bundle, its curvature, the corresponding parallel transport and holonomy. In Section 3 we review some basic ideas of singular chains and cochains which we then refine to what we call the *small complexes* that are needed in order to view the discrete connection form $\mathscr{A}_d$ and the discrete curvature $\mathscr{B}_d$ as singular 1 and 2 cochains $[\mathscr{A}_d]$ and $[\mathscr{B}_d]$ respectively. In Section 4 we obtain formulas to compute the discrete holonomy phase that, eventually, lead to the first version of our result expressing those phases in terms of integrals of the G-valued $[\mathscr{A}_d]$ and $[\mathscr{B}_d]$ (Theorem 4.3). Last, in Section 5 we consider $\mathfrak{g}$-valued singular cochains and construct logarithmic versions of $[\mathscr{A}_d]$ and $[\mathscr{B}_d]$, which appear in the expression of the discrete holonomy phase as an integral of $\mathfrak{g}$-valued cochains (Theorem 5.11).

[1] The basic motivation is that if time is discrete, a velocity vector —that is, an element of TQ— may be replaced by two nearby points —that is, an element of $Q \times Q$, near the diagonal Δ_Q.

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Notation: throughout this paper $\pi : Q \to M$ is a smooth principal (left) G-bundle — usually referred to as π in what follows— and we denote the (left) G-action on Q by $l^Q_g(q)$ for $q \in Q$ and $g \in G$. In addition to l^Q we will consider some other (left) G-actions: the diagonal action on $Q \times Q$ and the action on the first factor of $Q \times M$. Explicitly, for $g \in G$, $(q,q') \in Q \times Q$ and $(q,m) \in Q \times M$,

$$l^{Q\times Q}_g(q,q') := (l^Q_g(q), l^Q_g(q')) \quad \text{and} \quad l^{Q\times M}_g(q,m) := (l^Q_g(q), m).$$

We denote the *diagonal* of any Cartesian product $X \times X$ by Δ_X and the *discrete vertical submanifold* of Q by $\mathscr{V}_d := (\pi \times \pi)^{-1}(\Delta_M) \subset Q \times Q$.

2. Discrete connections on principal bundles

In this section we review the notion of discrete connection on a principal bundle (via discrete connection form and discrete horizontal lift), the associated curvature and the corresponding parallel transport.

2.1. **Discrete connection form and discrete horizontal lift.**

Discrete connections on a principal bundle can be constructed using different data. For this paper it will be sufficient to characterize discrete connections via their *discrete connection form* and their *discrete horizontal lift*. For more information on discrete connections see [FZ13].

Definition 2.1. An open subset $\mathscr{U} \subset Q \times Q$ is said to be of *D-type* if it is $G \times G$-invariant for the product of the G-action with itself and $\mathscr{V}_d \subset \mathscr{U}$ (in particular, $\Delta_Q \subset \mathscr{U}$).

Given a D-type subset $\mathscr{U} \subset Q \times Q$ we define

$$\mathscr{U}' := (id_Q \times \pi)(\mathscr{U}) \subset Q \times M \quad \text{and} \quad \mathscr{U}'' := (\pi \times \pi)(\mathscr{U}) \subset M \times M. \tag{2.1}$$

As π is a principal bundle map, both $\mathscr{U}'$ and $\mathscr{U}''$ are open subsets.

Definition 2.2. Let $\mathscr{U} \subset Q \times Q$ be a D-type subset. A smooth function $\mathscr{A}_d : \mathscr{U} \to G$ is called a *discrete connection form* on π if, for all $q \in Q$, $\mathscr{A}_d(q,q) = e$, the identity element of G, and it satisfies

$$\mathscr{A}_d(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = g_1 \mathscr{A}_d(q_0,q_1) g_0^{-1} \quad \text{for all} \quad (q_0,q_1) \in \mathscr{U}, \quad g_0, g_1 \in G.$$

Remark 2.3. An important difference between continuous and discrete connections on a principal G-bundle $\pi : Q \to M$ is that while the (continuous) connection 1-form is defined over all of TQ, a discrete connection over π is only defined in some open neighborhood of the diagonal $\Delta_Q \subset Q \times Q$, unless π is a trivial bundle. Indeed, if a discrete connection is globally defined, a global section of π can be readily constructed. For this reason, when dealing with discrete connections, their domain is very important.

Remark 2.4. Just as connections on principal bundles can be defined by a horizontal distribution, discrete principal connections can be defined by a *horizontal submanifold* $Hor_{\mathscr{A}_d} \subset Q \times Q$. Indeed, this is the approach of [LMW05] and [FZ13]. But, as shown in Theorem 3.6 of [FZ13], giving a horizontal submanifold is equivalent to giving a discrete connection form $\mathscr{A}_d$. The horizontal submanifold associated to $\mathscr{A}_d$ is $Hor_{\mathscr{A}_d} := {\mathscr{A}_d}^{-1}(\{e\})$.

Example 2.5. Let $\mathbb{R}_{>0} := (0,+\infty) \subset \mathbb{R}$ and $U(1) := \{z \in \mathbb{C} : |z| = 1\}$ seen as a multiplicative group under complex multiplication. Then, $\pi : Q \to M$ given by $p_1 : \mathbb{R}_{>0} \times U(1) \to \mathbb{R}_{>0}$ is a smooth (trivial) principal G-bundle, with $G := U(1)$ acting on the second factor by multiplication. Although it is not relevant for this work, the manifold Q can be seen, for instance, as the configuration manifold (for the center of mass description) of a planar mechanical system consisting of two equal-mass particles. For $\mu \in \mathbb{N}$, we define

$$\mathscr{A}_{d,\mu} : Q \times Q \to U(1) \quad \text{by} \quad \mathscr{A}_{d,\mu}((r_0,h_0),(r_1,h_1)) := \exp(i(r_1 - r_0)^\mu)\frac{h_1}{h_0}.$$

It is easy to check that $\mathscr{A}_{d,\mu}$ is a discrete connection form on π that is globally defined, *i.e.*, with domain $\mathscr{U} := Q \times Q$.

Definition 2.6. Let $\mathscr{U} \subset Q \times Q$ be of D-type. A smooth function $h_d : \mathscr{U}' \to Q \times Q$ is a *discrete horizontal lift* on π if the following conditions hold.

(1) $h_d : \mathscr{U}' \to Q \times Q$ is G-equivariant for the G-actions $l^{Q\times M}$ and $l^{Q\times Q}$.
(2) h_d is a section of $(id_Q \times \pi) : Q \times Q \to Q \times M$ over $\mathscr{U}'$, that is, $(id_Q \times \pi) \circ h_d = id_{\mathscr{U}'}$.
(3) For every $q \in Q$, $h_d(q, \pi(q)) = (q,q)$.

If $h_d : \mathscr{U}' \to Q \times Q$ is a discrete horizontal lift on π, we define $\overline{h_d} := p_2 \circ h_d$, where $p_2 : Q \times Q \to Q$ is the projection onto the second factor.

Discrete connection forms and discrete horizontal lifts are related as follows. First, recall the following construction: consider the fiber product $Q\ {}_\pi\!\times_\pi Q$ of π with itself —that is, the set of pairs (q_0,q_1) such that $\pi(q_0) = \pi(q_1)$. Let $\kappa : Q\ {}_\pi\!\times_\pi Q \to G$ be defined by $\kappa(q_0,q_1) := g$ if and only if $l_g^Q(q_0) = q_1$. It is easy to check that κ is a smooth function. Let $\mathscr{U} \subset Q \times Q$ be a D-type subset. Given a discrete connection form $\mathscr{A}_d : \mathscr{U} \to G$, we define

$$h_{\mathscr{A}_d} : \mathscr{U}' \to Q \times Q \quad \text{by} \quad h_{\mathscr{A}_d}(q,r) := (q, l^Q_{\mathscr{A}_d(q,q')^{-1}}(q')), \tag{2.2}$$

for any $q' \in Q|_r$. Conversely, given a discrete horizontal lift $h_d : \mathscr{U}' \to Q \times Q$, we define

$$\mathscr{A}_d^{h_d} : \mathscr{U} \to G \quad \text{by} \quad \mathscr{A}_d^{h_d}(q_0,q_1) := \kappa(\overline{h_d}(q_0,\pi(q_1)),q_1). \tag{2.3}$$

Theorem 2.7. *The maps $h_{\mathscr{A}_d}$ and $\mathscr{A}_d^{h_d}$ defined by (2.2) and (2.3) are a discrete horizontal lift and a discrete connection form on π respectively. Furthermore, the two constructions are inverses of each other.*

Proof. It follows from Theorems 3.6 and 4.6 of [FZ13]. □

As a consequence of Theorem 2.7 discrete connection forms and discrete horizontal lifts are alternative descriptions of a single object, the *discrete connections* with domain $\mathscr{U}$. In this spirit we speak of a "discrete connection" with domain $\mathscr{U}$ as the object defined by either one of these maps.

Remark 2.8. If $\mathscr{A}_d : \mathscr{U} \to G$ is a discrete connection form on π and $h_{\mathscr{A}_d}$ is the associated discrete horizontal lift, then $\mathscr{A}_d(h_{\mathscr{A}_d}(q,r)) = e$ for all $(q,r) \in \mathscr{U}'$.

Example 2.9. The discrete horizontal lift associated by Theorem 2.7 to the discrete connection form $\mathscr{A}_{d,\mu}$ introduced in Example 2.5 is, using (2.2),

$$h_{d,\mu} : (\mathbb{R}_{>0} \times U(1)) \times \mathbb{R}_{>0} \to (\mathbb{R}_{>0} \times U(1))^2 \quad \text{so that}$$
$$h_{d,\mu}((r_0,h_0),r_1) := ((r_0,h_0),(r_1,h_0\exp(-i(r_1-r_0)^\mu))).$$

Next we introduce a local description of discrete connection forms.

Definition 2.10. Let $V \subset M$ be an open set and $s : V \to Q$ a smooth section of $Q|_V$. Given a discrete connection $\mathscr{A}_d : \mathscr{U} \to G$ we define its *local expression* with respect to s by

$$\mathscr{A}_d^s : \mathscr{V}'' \to G \quad \text{so that} \quad \mathscr{A}_d^s(m_0,m_1) := \mathscr{A}_d(s(m_0),s(m_1)), \tag{2.4}$$

where $\mathscr{V}'' := (V \times V) \cap \mathscr{U}''$ (recall (2.1) for $\mathscr{U}''$).

Example 2.11. In the context of Example 2.5, the principal bundle is trivial. So we can take $V := \mathbb{R}_{>0}$ and a global section of p_1, $s(r) := (r,1)$. Then, the "local" expression of $\mathscr{A}_{d,\mu}$ is $\mathscr{A}^s_{d,\mu}(r_0,r_1) := \exp(i(r_1-r_0)^\mu)$ for all $(r_0,r_1) \in \mathscr{V}'' = M \times M = (\mathbb{R}_{>0})^2$.

The following properties of the local expression of a discrete connection are easy to check.

Lemma 2.12. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π and $s : V \to Q$ be a section of $Q|_V$. Then, $\mathscr{V}'' \subset M \times M$ is open, $\mathscr{A}^s_d$ is smooth and $\mathscr{A}^s_d(m,m) = e$ for all $m \in V$. If $\varphi_s : V \times G \to Q$ is defined by $\varphi_s(m,g) := l^Q_g(s(m))$, then $\mathscr{A}_d(\varphi_s(m_0,g_0),\varphi_s(m_1,g_1)) = g_1 \mathscr{A}^s_d(m_0,m_1) g_0^{-1}$.*

2.2. **Curvature of a discrete connection.** The curvature of a connection on a principal bundle can be seen as an obstruction to the local trivializability of the bundle (with connection) or, alternatively, as the obstruction to a certain map appearing in the Atiyah sequence being a morphism of Lie algebroids. In a similar manner, a notion of curvature of a discrete connection on a principal bundle can be introduced as the obstruction to the bundle (with discrete connection) being locally trivializable (see [FZ]) or, alternatively, to a certain map appearing in the discrete Atiyah sequence being a morphism of local Lie groupoids (see [FJZ22]).

Definition 2.13. Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection form on π. The *curvature* of $\mathscr{A}_d$ is the map

$$\mathscr{B}_d : \mathscr{U}^{(3)} \to G \quad \text{so that} \quad \mathscr{B}_d(q_0,q_1,q_2) := \mathscr{A}_d(q_0,q_2)^{-1} \mathscr{A}_d(q_1,q_2) \mathscr{A}_d(q_0,q_1),$$

where $\mathscr{U}^{(3)} := \{(q_0,q_1,q_2) \in Q^3 : (q_j,q_k) \in \mathscr{U} \text{ for all } 0 \le j < k \le 2\}$. A discrete connection form is *flat* if its curvature is constantly e.

When $V \subset M$ is an open set and $s : V \to Q$ is a smooth section of $Q|_V$ we define the *local expression of the curvature* with respect to s by

$$\mathscr{B}^s_d : \mathscr{V}''^{(3)} \to G \quad \text{so that} \quad \mathscr{B}^s_d(m_0,m_1,m_2) := \mathscr{B}_d(s(m_0),s(m_1),s(m_2)), \tag{2.5}$$

where

$$\mathscr{V}''^{(3)} := \{(m_0,m_1,m_2) \in M^3 : (m_j,m_k) \in \mathscr{V}'' \text{ for all } 0 \le j < k \le 2\}. \tag{2.6}$$

Example 2.14. The curvature of the discrete connection $\mathscr{A}_{d,\mu}$ introduced in Example 2.5 is

$$\mathscr{B}_{d,\mu}((r_0,h_0),(r_1,h_1),(r_2,h_2)) = \exp(i(-(r_2-r_0)^\mu + (r_2-r_1)^\mu + (r_1-r_0)^\mu)),$$

for all $((r_0,h_0),(r_1,h_1),(r_2,h_2)) \in \mathscr{U}^{(3)} = Q^3$. Also, in the local trivialization considered in Example 2.11,

$$\mathscr{B}^s_{d,\mu}(r_0,r_1,r_2) = \exp(i(-(r_2-r_0)^\mu + (r_2-r_1)^\mu + (r_1-r_0)^\mu)), \tag{2.7}$$

for all $(r_0,r_1,r_2) \in \mathscr{V}''^{(3)} = (R_{>0})^3$. It is easy to check that $\mathscr{B}_{d,\mu} = 1$ if and only if $\mu = 1$.

The following properties of the local expression of the curvature of a discrete connection are easy to check.

Lemma 2.15. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π and $s : V \to Q$ a section of $Q|_V$. Then, $\mathscr{B}^s_d$ is smooth and $\mathscr{B}^s_d(m_0,m_1,m_2) = \mathscr{A}^s_d(m_0,m_2)^{-1} \mathscr{A}^s_d(m_1,m_2) \mathscr{A}^s_d(m_0,m_1)$ for all $(m_0,m_1,m_2) \in \mathscr{V}''^{(3)}$. Also, $\mathscr{B}_d(\varphi_s(m_0,g_0),\varphi_s(m_1,g_1),\varphi_s(m_2,g_2)) = g_0 \mathscr{B}^s_d(m_0,m_1,m_2) g_0^{-1}$ for all $(m_0,m_1,m_2) \in \mathscr{V}''^{(3)}$ and $g_0,g_1,g_2 \in G$, where φ_s is defined in Lemma 2.12.*

2.3. **Parallel transport associated to a discrete connection.** Given a set X and $N \in \mathbb{N} \cup \{0\}$, a *discrete path* of length N is an element $x_. := (x_0, \ldots, x_N) \in X^{N+1}$. The initial and final points of $x_.$ are x_0 and x_N respectively. When $x_0 = x_N$, $x_.$ is a discrete N-*loop*. The set of all discrete paths of length N with initial point $\overline{x}$ and final point $\overline{x}'$ is denoted by $\Omega_N(\overline{x}, \overline{x}')$ and the discrete N-loops with initial point $\overline{x}$ is denoted by $\Omega_N(\overline{x})$. Last, we define $\Omega(\overline{x}, \overline{x}') := \cup_{N=0}^{\infty} \Omega_N(\overline{x}, \overline{x}')$ and $\Omega(\overline{x}) := \Omega(\overline{x}, \overline{x})$.

In what follows, it will be necessary to consider discrete paths that satisfy some restrictions. Thus, given $U \subset X \times X$ and $N \in \mathbb{N} \cup \{0\}$ we define the following sets of discrete paths *subordinated* to U: $\Omega_{N,U}(\overline{x}, \overline{x}') := \{x_. \in \Omega_N(\overline{x}, \overline{x}') : (x_{k-1}, x_k) \in U \text{ for all } k = 1, \ldots, N\}$, $\Omega_{N,U}(\overline{x}) := \Omega_{N,U}(\overline{x}, \overline{x})$, $\Omega_U(\overline{x}, \overline{x}') := \cup_{N=0}^{\infty} \Omega_{N,U}(\overline{x}, \overline{x}')$ and $\Omega_U(\overline{x}) := \Omega_U(\overline{x}, \overline{x})$.

Let $\mathscr{U} \subset Q \times Q$ be a D-type subset and $h_d : \mathscr{U}' \to Q \times Q$ be a discrete horizontal lift on the principal bundle π. Then, for any discrete path in M, $m_. \in \Omega_{N,\mathscr{U}''}(\overline{m}, \overline{m}')$ and $\overline{q} \in Q|_{\overline{m}}$ we define inductively $q_0 := \overline{q}$ and, for each $k = 1, \ldots, N$, $q_k := \overline{h_d}(q_{k-1}, m_k)$. It is easy to verify that, as $(m_{k-1}, m_k) \in \mathscr{U}''$ for all $k = 1, \ldots, N$, all $(q_{k-1}, m_k) \in \mathscr{U}'$, so that each q_k is well defined and $\pi(q_k) = m_k$. Let $q_. := (q_0, \ldots, q_N) \in \Omega_N(\overline{q}, q_N)$; by construction, $\pi(q_N) = m_N = \overline{m}'$. The discrete path $q_.$ is called the *discrete horizontal lift* of $m_.$ starting at $\overline{q}$. This leads to the following notion.

Definition 2.16. Let $\mathscr{U} \subset Q \times Q$ be a D-type subset and $h_d : \mathscr{U}' \to Q \times Q$ be a discrete horizontal lift on π. For each $m_. \in \Omega_{N,\mathscr{U}''}(\overline{m}, \overline{m}')$ we define the *discrete parallel transport map* over $m_.$

$$\mathrm{PT}_d : Q|_{\overline{m}} \to Q|_{\overline{m}'} \quad \text{ so that } \quad \mathrm{PT}_d(m_.)(\overline{q}) := q_N,$$

where q_N is the one constructed in the previous paragraph.

Of interest for our analysis is the special case where one considers parallel transport over discrete loops, so that PT_d is a map from a fiber of Q onto itself.

Definition 2.17. Let $\mathscr{U} \subset Q \times Q$ be a D-type subset and $h_d : \mathscr{U}' \to Q \times Q$ be a discrete horizontal lift on π. For each $m_. \in \Omega_{N,\mathscr{U}''}(\overline{m})$ and $\overline{q} \in Q|_{\overline{m}}$ we define the *discrete holonomy phase* around $m_.$ starting at $\overline{q}$ as

$$\Phi_d(m_., \overline{q}) := \kappa(\overline{q}, \mathrm{PT}_d(m_.)(\overline{q})) \in G.$$

The discrete holonomy phase is well defined because $\pi(\mathrm{PT}_d(m_.)(\overline{q})) = \pi(q_N) = m_N = \overline{m} = \pi(\overline{q})$.

Example 2.18. The parallel transport operator associated to the discrete connection form $\mathscr{A}_{d,\mu}$ defined in Example 2.5 is constructed as follows. For $\overline{r}, \overline{r}' \in \mathbb{R}_{>0}$ fix a discrete path in $\mathbb{R}_{>0}$, $r_. \in \Omega_N(\overline{r}, \overline{r}')$ —we ignore $\mathscr{U}''$ because $\mathscr{A}_{d,\mu}$ is globally defined— and choose $\overline{q} := (\overline{r}, \overline{h}) \in Q|_{\overline{r}}$. Then, using $h_{d,\mu}$ computed in Example 2.9, the discrete lifted path of $r_.$ starting at $\overline{q}$ is given by

$$q_k = \begin{cases} \overline{q}, & \text{if } \ k = 0, \\ (r_k, \overline{h} \exp(-i \sum_{j=1}^{k} (r_j - r_{j-1})^{\mu})), & \text{if } \ k = 1, \ldots, N. \end{cases}$$

Therefore $\mathrm{PT}_d(\overline{r}, \overline{h}) = (\overline{r}', (r_k, \overline{h} \exp(-i \sum_{j=1}^{N} (r_j - r_{j-1})^{\mu})))$ and

$$\Phi_d(r_., (\overline{r}, \overline{h})) = \exp\left(-i \sum_{j=1}^{N} (r_j - r_{j-1})^{\mu}\right).$$

A natural question is how the discrete holonomy phase $\Phi_d(m_., \overline{q})$ changes when $\overline{q}$ is replaced by $l_g^Q(\overline{q})$.

Proposition 2.19. *Let $h_d : \mathscr{U}' \to Q \times Q$ be a discrete horizontal lift on π, $\overline{m} \in M$, $\overline{q} \in Q|_{\overline{m}}$ and $m_\cdot \in \Omega_N(\overline{m})$. Then, for any $g \in G$,*

1. *If $q_\cdot \in \Omega_N(\overline{q}, q_N)$ is the discrete horizontal lift of the path $m_\cdot$ starting at $\overline{q}$, the path $q'_\cdot$ defined by $q'_k := l^Q_g(q_k)$ for $k = 0, \ldots, N$ is the discrete horizontal lift of $m_\cdot$ starting at $l^Q_g(\overline{q})$.*
2. *In addition, $\Phi_d(m_\cdot, l^Q_g(\overline{q})) = g\,\Phi_d(m_\cdot, \overline{q})g^{-1}$.*

Proof. By definition, $q'_\cdot \in \Omega_N(l^Q_g(\overline{q}), l^Q_g(q_N))$. That $q'_\cdot$ is the discrete horizontal lift of $m_\cdot$ starting at $l^Q_g(\overline{q})$ follows by the definition of discrete horizontal lift and property (1) in Definition 2.6. Thus,

$$\begin{aligned}\Phi_d(m_\cdot, l^Q_g(\overline{q})) &= \kappa(l^Q_g(\overline{q}), \mathrm{PT}_d(m_\cdot)(l^Q_g(\overline{q}))) = \kappa(l^Q_g(\overline{q}), l^Q_g(\mathrm{PT}_d(m_\cdot)(\overline{q})))\\ &= g\kappa(\overline{q}, \mathrm{PT}_d(m_\cdot)(\overline{q}))g^{-1} = g\,\Phi_d(m_\cdot, \overline{q})g^{-1}.\end{aligned}$$

□

The following two local results will allow us, later, to find explicit formulas relating the holonomy phase and the curvature of a discrete connection.

Lemma 2.20. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection form on π, $V \subset M$ be an open subset and $s : V \to Q$ be a section of π. Then, for each $m_\cdot \in \Omega_{N,\mathscr{V}''}(m_0, m_N)$ (with $\mathscr{V}''$ as in Definition 2.10), the following assertions are true.*

1. *Let $q_\cdot \in \Omega_N(q_0, q_N)$ be the discrete horizontal lift path of $m_\cdot$ starting at $q_0 \in Q|_{m_0}$. Then, there are unique $g_k \in G$ so that, for φ_s as in Lemma 2.12, $q_k = \varphi_s(m_k, g_k)$ for all $k = 0, \ldots, N$.*
2. *We have $\mathrm{PT}_d(m_\cdot)(\varphi_s(m_0, g_0)) = \varphi_s(m_N, g_N)$ for $g_N := g_0 \prod_{k=1}^N \mathscr{A}^s_d(m_{k-1}, m_k)^{-1}$.*

Proof. As $\pi(q_k) = m_k$ and $(m_{k-1}, m_k) \in \mathscr{V}'' \subset V \times V$ for all k, we see that $q_k \in Q|_V$. Point (1) follows immediately because φ_s trivializes $Q|_V$.

By definition of PT_d and point (1) we have that $\mathrm{PT}_d(m_\cdot)(\varphi_s(m_0, g_0)) = \varphi_s(m_N, g_N)$. Then, as $(q_{k-1}, q_k) = h_{\mathscr{A}_d}(q_{k-1}, m_k)$ and $\varphi_s(m, g) = l^Q_g(s(m))$, recalling Remark 2.8, we have

$$\begin{aligned}e &= \mathscr{A}_d(q_{k-1}, q_k) = \mathscr{A}_d(\varphi_s(m_{k-1}, g_{k-1}), \varphi_s(m_k, g_k))\\ &= \mathscr{A}_d(l^Q_{g_{k-1}}(s(m_{k-1})), l^Q_{g_k}(s(m_k))) = g_k \mathscr{A}^s_d(m_{k-1}, m_k) g^{-1}_{k-1}.\end{aligned}$$

Thus, $g_k = g_{k-1}\mathscr{A}^s_d(m_{k-1}, m_k)^{-1}$ for all k. Point (2) follows by applying this formula recursively. □

Proposition 2.21. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π. Then, for each $N \geq 2$, the following statements are true.*

1. *For each $q_\cdot \in \Omega_N(\overline{q}, \overline{q}')$ such that $(q_0, q_k, q_{k+1}) \in \mathscr{U}^{(3)}$ for all $k = 1, \ldots, N-1$ we have*
$$\left(\prod_{k=1}^N \mathscr{A}_d(q_{k-1}, q_k)^{-1}\right)\mathscr{A}_d(q_0, q_N) = \prod_{k=1}^{N-1} \mathscr{B}_d(q_0, q_k, q_{k+1})^{-1}.$$
2. *Let $V \subset M$ be an open subset and $s : V \to Q$ be a section of Q. Then, for each $m_\cdot \in \Omega_N(\overline{m})$ such that $(m_0, m_k, m_{k+1}) \in \mathscr{V}''^{(3)}$ (see (2.6)) for all $k = 0, \ldots, N-1$ and for each $\overline{q} \in Q|_{\overline{m}}$ we have*
$$\Phi_d(m_\cdot, \overline{q}) = g_0 \left(\prod_{k=1}^N \mathscr{A}^s_d(m_{k-1}, m_k)^{-1}\right) g_0^{-1} = \prod_{k=1}^{N-1} \mathscr{B}_d(q_0, q_k, q_{k+1})^{-1}, \tag{2.8}$$
where $q_\cdot \in \Omega_N(\overline{q})$ is the discrete horizontal lift of $m_\cdot$ and g_0 satisfies $\varphi_s(m_0, g_0) = \overline{q}$.

Proof. Point (1) can be readily checked by induction (on N). In order to prove point (2) we notice that, since $(m_0, m_k, m_{k+1}) \in \mathscr{V}''^{(3)}$ for all $k = 0, \dots, N-1$, it follows that $m_. \in \Omega_{N,\mathscr{V}''}(\overline{m})$. Hence, by Lemma 2.20 there exists the discrete horizontal lift $q_.$ of $m_.$ and, writing $q_k = \varphi_s(m_k, g_k)$ for all k, it satisfies $\mathrm{PT}_d(m_.)(\varphi_s(m_0, g_0)) = \varphi_s(m_N, g_N)$ for $g_N = g_0 \prod_{k=1}^N \mathscr{A}_d^s(m_{k-1}, m_k)^{-1}$. Then,

$$\begin{aligned}\Phi_d(m_., \overline{q}) &= \kappa(q_0, \mathrm{PT}_d(m_.)(\varphi_s(m_0, g_0))) = \kappa(\varphi_s(m_0, g_0), \varphi_s(\underbrace{m_N}_{=m_0}, g_N)) \\ &= \kappa(l^Q_{g_0}(s(m_0)), l^Q_{g_N}(s(m_0))) = g_N g_0^{-1} = g_0 \prod_{k=1}^N \mathscr{A}_d^s(m_{k-1}, m_k)^{-1} g_0^{-1},\end{aligned}$$

proving the first equality of (2.8). In order to prove the second equality, it is easy to check that, as $(m_0, m_k, m_{k+1}) \in \mathscr{V}''^{(3)}$ for all k, we have $(s(m_0), s(m_k), s(m_{k+1})) \in \mathscr{U}^{(3)}$ for all k. Then, using the previous computation and the result of point (1) we have

$$\begin{aligned}\Phi_d(m_., \overline{q}) &= g_0 \left(\prod_{k=1}^N \mathscr{A}_d^s(m_{k-1}, m_k)^{-1} \right) g_0^{-1} = g_0 \left(\prod_{k=1}^N \mathscr{A}_d(s(m_{k-1}), s(m_k))^{-1} \right) g_0^{-1} \\ &= g_0 \left(\prod_{k=1}^{N-1} \mathscr{B}_d(s(m_0), s(m_k), s(m_{k+1}))^{-1} \right) \underbrace{\mathscr{A}_d(s(m_0), s(\overbrace{m_N}^{=m_0}))^{-1}}_{=e} g_0^{-1} \\ &= g_0 \left(\prod_{k=1}^{N-1} \mathscr{B}_d^s(m_0, m_k, m_{k+1})^{-1} \right) g_0^{-1}.\end{aligned}$$

The second equality of (2.8) now follows from Lemma 2.15. □

Remark 2.22. By (2.8), we see that all the discrete holonomy phases (for loops satisfying the conditions of point (2) of Proposition 2.21) are products of values of the (inverse of the) curvature of the discrete connection. In this sense, this result can be seen as a seminal observation towards a discrete analogue of the Ambrose–Singer Theorem.

3. Singular (co)homology and adaptations

In this section we review some basic standard notions used in singular homology theory (see, for instance, [Mun84]) and, then, make an adaptation to have a theory that works with discrete connection forms and curvatures.

Let X be a topological space, $n \in \mathbb{N} \cup \{0\}$ and $\{e_0, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$[2].

Definition 3.1. Given $n \in \mathbb{N} \cup \{0\}$, the set

$$\Delta_n := \left\{ \sum_{j=0}^n t_j e_j \in \mathbb{R}^{n+1} : t_j \geq 0 \text{ for } j = 0, \dots, n \text{ and } \sum_{j=0}^n t_j = 1 \right\}$$

is called the *standard n-simplex*. A *singular n-simplex* of X is a continuous map $T_n : \Delta_n \to X$. A *singular n-chain* of X is an element of the free abelian group generated by the singular n-simplexes, that we denote by $S_n(X)$. When $n \in \mathbb{N}$ and $k = 0, \dots, n$, the k-th face of the

[2]In this context, it is convenient to consider $\mathbb{R}^n \subset \mathbb{R}^{\mathbb{N}}$ as the subset consisting of sequences vanishing after the n-th component. Thus, $\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^n$ and the inclusion a linear map.

standard n-simplex is the map $\partial_n^k : \Delta_{n-1} \to \Delta_n$ defined by

$$\partial_n^k \left(\sum_{j=0}^{n-1} t_j e_j \right) := \sum_{j=0}^{k-1} t_j e_j + \sum_{j=k+1}^{n} t_{j-1} e_j, \quad \text{if} \quad k = 0, \ldots, n.$$

For $n \in \mathbb{N}$ the *n-th boundary map* is the group homomorphism $\partial_n : S_n(X) \to S_{n-1}(X)$ defined by

$$\partial_n(T) := \sum_{k=0}^{n} (-1)^k (T \circ \partial_n^k).$$

We observe that, for $n \in \mathbb{N}$ and $e_j \in \Delta_{n-1}$, we have

$$\partial_n^k(e_j) = \begin{cases} e_{j+1} \in \Delta_n, & \text{if} \quad k = 0, \ldots, j, \\ e_j \in \Delta_n, & \text{if} \quad k = j+1, \ldots, n. \end{cases}$$

Example 3.2. Let T_1 be a singular 1-simplex of X, then $\partial_1 T_1 \in S_0(X)$ satisfies $(\partial_1 T_1)(1) = (T_1 \circ \partial_1^0)(1) - (T_1 \circ \partial_1^1)(1) = T_1(0,1) - T_1(1,0)$. Similarly, if T_2 is a singular 2-simplex of X, then $\partial_2 T_2 \in S_1(X)$ satisfies $(\partial_2 T_2)(t_0,t_1) = (T_2 \circ \partial_2^0)(t_0,t_1) - (T_2 \circ \partial_2^1)(t_0,t_1) + (T_2 \circ \partial_2^2)(t_0,t_1) = T_2(0,t_0,t_1) - T_2(t_0,0,t_1) + T_2(t_0,t_1,0)$.

Definition 3.3. Let A be an abelian group and $n \in \mathbb{N} \cup \{0\}$. The group of *singular n-cochains* of X is $S^n(X,A) := \hom(S_n(X),A)$ and the group homomorphism $\delta_n : S^n(X,A) \to S^{n+1}(X,A)$ defined by $\delta_n \alpha_n := \alpha_n \circ \partial_{n+1}$ is the *singular n-coboundary* of X.

Example 3.4. Let $\alpha_0 \in S^0(X,A)$ and $\alpha_1 \in S^1(X,A)$; using the computations from Example 3.2, for any singular 1-simplex T_1 of X we have

$$(\delta_0 \alpha_0)(T_1) = \alpha_0(\partial_1 T_1) = \alpha_0(T_1(0,1) - T_1(1,0)) = \alpha_0(T_1(0,1)) - \alpha_0(T_1(1,0)).$$

Analogously, for any singular 2-simplex T_2 of X we have

$$\begin{aligned}(\delta_1 \alpha_1)(T_2) &= \alpha_1((\partial_2 T_2)(t_0,t_1)) = \alpha_1(T_2(0,t_0,t_1) - T_2(t_0,0,t_1) + T_2(t_0,t_1,0)) \\ &= \alpha_1(T_2(0,t_0,t_1)) - \alpha_1(T_2(t_0,0,t_1)) + \alpha_2(T_2(t_0,t_1,0)).\end{aligned}$$

Notice that in this example we denote the group operations in $S_n(X)$, $S^n(X,A)$ and A additively. Later on we may use the product notation for the group operation in A and $S^n(X,A)$.

Remark 3.5. It is easy to check that $\partial_{n-1} \circ \partial_n = 0$ for all $n \geq 2$ and, consequently, that $\delta_{n+1} \circ \delta_n = 0$ for all $n \geq 0$.

Definition 3.6. We denote the duality pairing between $S_n(X)$ and $S^n(X,A)$ by $\int$. Thus, for any $\alpha_n \in S^n(X,A)$ and $T_n \in S_n(X)$ we have

$$\int_{T_n} \alpha_n := \alpha_n(T_n).$$

Remark 3.7. For any $T_{n+1} \in S_{n+1}(X)$ and $\alpha_n \in S^n(X,A)$ we have

$$\int_{\partial_{n+1} T_{n+1}} \alpha_n = \alpha_n(\partial_{n+1} T_{n+1}) = (\delta_n \alpha_n)(T_{n+1}) = \int_{T_{n+1}} \delta_n \alpha_n, \tag{3.1}$$

that is, the "singular Stokes' Theorem".

3.1. **Small singular chains and cochains.** Next we want to consider a small variation of the previous construction, with the goal of defining chains and cochains of X that are, in a sense, controlled by an open set.

Let X be a topological space and $U \subset X \times X$ be an open subset (for the product topology). Then, for any $n \in \mathbb{N}$ we define

$$U^{(n+1)} := \{(x_0, x_1, \ldots, x_n) \in X^{n+1} : (x_j, x_k) \in U \text{ for all } 0 \leq j < k \leq n\}. \tag{3.2}$$

Definition 3.8. For $n \in \mathbb{N}$, a singular n-simplex T_n of X is said to be U*-small* (*small* if no confusion arises) if $(T_n(e_0), \ldots, T_n(e_n)) \in U^{(n+1)}$. The free abelian group generated by the U-small singular n-simplexes of X is denoted by $S_{n,U}(X)$ and its elements will be called U*-small singular n-chains* of X. For completeness, we define $S_{0,U}(X) := S_0(X)$.

The following result is straightforward.

Lemma 3.9. *For each $n \geq 0$, $S_{n,U}(X) \subset S_n(X)$ is a subgroup and, for each $n \geq 1$, the map $\partial_n^U : S_{n,U}(X) \to S_{n-1,U}(X)$ defined as the restriction and co-restriction of ∂_n is a homomorphism satisfying $\partial_n^U \circ \partial_{n+1}^U = 0$.*

Definition 3.10. Let A be an abelian group and $U \subset X \times X$ an open subset. For each $n \in \mathbb{N} \cup \{0\}$ we define the group $S^{n,U}(X,A) := \hom(S_{n,U}(X), A)$; its elements are called U*-small singular cochains of* X. The group homomorphism $\delta_n^U : S^{n,U}(X,A) \to S^{n+1,U}(X,A)$ defined by $\delta_n^U \alpha_n := \alpha_n \circ \partial_{n+1}^U$ is the *small singular n-coboundary* of X.

The following result is straightforward.

Lemma 3.11. *For each $n \in \mathbb{N} \cup \{0\}$, $S^n(X,A) \subset S^{n,U}(X,A)$ is a subgroup and the homomorphism δ_n^U restricted and co-restricted appropriately coincides with δ_n. Furthermore, $\delta_{n+1}^U \circ \delta_n^U = 0$.*

Last we have a simple "change of coefficients" relation. Let $f \in \hom(A, A')$, where A and A' are abelian groups. It is easy to check that the map $f_* : S^{n,U}(X,A) \to S^{n,U}(X,A')$ defined by $f_*(\alpha_n) := f \circ \alpha_n$ is a homomorphism satisfying $f_* \circ \delta_n^{U,A} = \delta_{n+1}^{U,A'} \circ f_*$ (*i.e.*, f_* is a homomorphism of cochain complexes).

As an application of the ideas developed so far, in the next section we construct small singular cochains associated to a discrete connection on π, when the structure group of the principal bundle is abelian.

3.2. **Discrete connections and singular cochains.** Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π and assume that the structure group G of π is abelian. Let $T_1 \in S_{1,\mathscr{U}}(Q)$ be a 1-simplex. Then, we define

$$[\mathscr{A}_d](T_1) := \mathscr{A}_d(T_1(1,0), T_1(0,1)) \in G.$$

Being these 1-simplexes free generators of $S_{1,\mathscr{U}}(Q)$, this assignment defines a unique element $[\mathscr{A}_d] \in S^{1,\mathscr{U}}(Q,G)$.

Observe that, if Q is connected, the cochain $[\mathscr{A}_d]$ determines the function $\mathscr{A}_d : \mathscr{U} \to G$. Thus, the discrete connection form of a discrete connection is a true 1-form (a cochain) for this (small) singular cohomology, at least when G is abelian. If $\mathscr{B}_d : \mathscr{U}^{(3)} \to G$ is the curvature of $\mathscr{A}_d$ and $T_2 \in S_{2,\mathscr{U}}(Q,G)$ is a 2-simplex, we define

$$[\mathscr{B}_d](T_2) := \mathscr{B}_d(T_2(1,0,0), T_2(0,1,0), T_2(0,0,1)) \in G.$$

As those simplexes freely generate $S_{2,\mathscr{U}}(Q)$, this formula determines a unique $[\mathscr{B}_d] \in S^{2,\mathscr{U}}(Q,G)$.

Example 3.12. In the case of the discrete connection form $\mathscr{A}_{d,\mu}$ introduced in Example 2.5, if $T_1 = \sum_{j=1}^N a_j T_1^j \in S_{1,Q\times Q}(\mathbb{R}_{>0}\times U(1)) = S_1(\mathbb{R}_{>0}\times U(1))$, using multiplicative notation, we have

$$[\mathscr{A}_{d,\mu}](T_1) = \prod_{j=1}^N \mathscr{A}_{d,\mu}(\underbrace{T_1^j(1,0)}_{=:(r_0^j,h_0^j)}, \underbrace{T_1^j(0,1)}_{=:(r_1^j,h_1^j)})^{a_j} = \exp\left(i\sum_{j=1}^N a_j(r_1^j - r_0^j)^\mu\right)\prod_{j=1}^N \left(\frac{h_1^j}{h_0^j}\right)^{a_j}.$$

Similarly, if $T_2 = \sum_{j=1}^N a_j T_2^j \in S_{2,Q\times Q}(\mathbb{R}_{>0}\times U(1)) = S_2(\mathbb{R}_{>0}\times U(1))$, we have

$$\begin{aligned}[\mathscr{B}_{d,\mu}](T_2) &= \prod_{j=1}^N \mathscr{B}_{d,\mu}(\underbrace{T_2^j(1,0,0)}_{=:(r_0^j,h_0^j)}, \underbrace{T_2^j(0,1,0)}_{=:(r_1^j,h_1^j)}, \underbrace{T_2^j(0,0,1)}_{=:(r_2^j,h_2^j)})^{a_j}\\ &= \exp\left(i\sum_{j=1}^N a_j\left(-(r_2^j - r_0^j)^\mu + (r_2^j - r_1^j)^\mu + (r_1^j - r_0^j)^\mu\right)\right).\end{aligned}$$

Proposition 3.13. *For $[\mathscr{A}_d] \in S^{1,\mathscr{U}}(Q,G)$ and $[\mathscr{B}_d] \in S^{2,\mathscr{U}}(Q,G)$ as above, we have $[\mathscr{B}_d] = \delta_1^{\mathscr{U}}[\mathscr{A}_d]$ and $\delta_2^{\mathscr{U}}[\mathscr{B}_d] = e$.*

Proof. The second assertion follows from the first and the fact that $\delta_2^{\mathscr{U}} \circ \delta_1^{\mathscr{U}} = e$. As the 2-simplexes generate $S_{2,\mathscr{U}}(Q)$, it suffices to check the first assertion of the statement on them to conclude its validity in general. Let T_2 be one such 2-simplex. Then,

$$\begin{aligned}[\mathscr{B}_d](T_2) &= \mathscr{B}_d(T_2(1,0,0),T_2(0,1,0),T_2(0,0,1))\\ &= \mathscr{A}_d(T_2(1,0,0),T_2(0,0,1))^{-1}\mathscr{A}_d(T_2(0,1,0),T_2(0,0,1))\mathscr{A}_d(T_2(1,0,0),T_2(0,1,0)).\end{aligned}$$

On the other hand, converting to multiplicative notation the computations of Example 3.4,

$$\begin{aligned}&(\delta_1^{\mathscr{U}}[\mathscr{A}_d])(T_2)\\ &\quad= [\mathscr{A}_d](T_2(0,t_0,t_1))([\mathscr{A}_d](T_2(t_0,0,t_1)))^{-1}[\mathscr{A}_d](T_2(t_0,t_1,0))\\ &\quad= \mathscr{A}_d(T_2(0,1,0),T_2(0,0,1))\mathscr{A}_d(T_2(1,0,0),T_2(0,0,1))^{-1}\mathscr{A}_d(T_2(1,0,0),T_2(0,1,0)).\end{aligned}$$

The result follows by comparing the two formulas. ☐

Remark 3.14. The expression $\delta_2^{\mathscr{U}}[\mathscr{B}_d] = e$ may be interpreted as a discrete version of Bianchi's Identity.

A nice immediate consequence of the formalism developed is the following result.

Corollary 3.15. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on $\pi : Q \to M$ with Q connected. If there is $\alpha_0 \in S^0(Q)$ such that $[\mathscr{A}_d] = \delta_0^{\mathscr{U}}\alpha_0$ then $\mathscr{A}_d$ is flat, i.e., $\mathscr{B}_d = e$.*

Proof. As $[\mathscr{A}_d] = \delta_0^{\mathscr{U}}\alpha_0$, by Proposition 3.13 and Lemma 3.11, we have $[\mathscr{B}_d] = \delta_1^{\mathscr{U}}[\mathscr{A}_d] = (\delta_1^{\mathscr{U}} \circ \delta_0^{\mathscr{U}})\alpha_0 = e$. Hence, for any $T_2 \in S_{2,\mathscr{U}}(Q)$,

$$e = [\mathscr{B}_d](T_2) = \mathscr{B}_d(T_2(1,0,0),T_2(0,1,0),T_2(0,0,1)). \tag{3.3}$$

For any $(q_0,q_1,q_2) \in \mathscr{U}^{(3)}$, as Q is connected (hence, path connected), there is a continuous map $\gamma\colon [0,1] \to Q$ such that $\gamma(0) = q_0$, $\gamma(\frac{1}{2}) = q_1$ and $\gamma(1) = q_2$. Let $T_2 : \Delta_2 \to Q$ be defined by $T_2(t_0,t_1,t_2) := \gamma(\frac{1}{2}t_1 + t_2)$; it is immediate that $T_2 \in S_{2,\mathscr{U}}(Q)$ and it satisfies $T_2(e_j) = q_j$ for $j = 0,1,2$. All together, using (3.3),

$$\mathscr{B}_d(q_0,q_1,q_2) = \mathscr{B}_d(T_2(1,0,0),T_2(0,1,0),T_2(0,0,1)) = e.$$

☐

3.3. **The local version.** Just as we described in Section 3.2 how to view discrete connection forms and the corresponding curvature of a principal bundle as (small) singular cochains, it is possible to do the same for the local expressions of those objects, as we discuss below.

Let $V \subset M$ be an open subset, $s : V \to Q$ be a section of π and $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection form on π. Let $\mathscr{A}_d^s : \mathscr{V}'' \to G$ be the corresponding local expression of $\mathscr{A}_d$ with respect to s (2.4). For each 1-simplex $T_1 \in S_{1,\mathscr{V}''}(M)$ we define

$$[\mathscr{A}_d^s](T_1) := \mathscr{A}_d^s(T_1(1,0), T_1(0,1)) \in G.$$

Being those simplexes generators of $S_{1,\mathscr{V}''}(M)$, this assignment determines a unique element $[\mathscr{A}_d^s] \in S^{1,\mathscr{V}''}(M,G)$.

Similarly, let $\mathscr{B}_d^s : \mathscr{V}''^{(3)} \to G$ be the local expression of the curvature of $\mathscr{A}_d$ (2.5) and $T_2 \in S_{2,\mathscr{V}''}(M)$ be a 2-simplex. We define

$$[\mathscr{B}_d^s](T_2) := \mathscr{B}_d^s(T_2(1,0,0), T_2(0,1,0), T_2(0,0,1)) \in G.$$

Just as before, as the 2-simplexes generate $S_{2,\mathscr{V}''}(M)$ this last expression defines a unique element $[\mathscr{B}_d^s] \in S^{2,\mathscr{V}''}(M,G)$.

Example 3.16. In the case of the discrete connection form $\mathscr{A}_{d,\mu}$ introduced in Example 2.5 together with the (global) trivialization of Example 2.11, if $T_1 = \sum_{j=1}^N a_j T_1^j \in S_{1,M\times M}(\mathbb{R}_{>0}) = S_1(\mathbb{R}_{>0})$, we have

$$[\mathscr{A}_{d,\mu}^s](T_1) = \prod_{j=1}^N \mathscr{A}_{d,\mu}^s(\underbrace{T_1^j(1,0)}_{=:r_0^j}, \underbrace{T_1^j(0,1)}_{=:r_1^j})^{a_j} = \exp\left(i\sum_{j=1}^N a_j (r_1^j - r_0^j)^\mu\right).$$

Similarly, if $T_2 = \sum_{j=1}^N a_j T_2^j \in S_{2,M\times M}(\mathbb{R}_{>0}) = S_2(\mathbb{R}_{>0})$, we have

$$\begin{aligned}[\mathscr{B}_{d,\mu}^s](T_2) &= \prod_{j=1}^N \mathscr{B}_{d,\mu}^s(\underbrace{T_2^j(1,0,0)}_{=:r_0^j}, \underbrace{T_2^j(0,1,0)}_{=:r_1^j}, \underbrace{T_2^j(0,0,1)}_{=:r_2^j})^{a_j} \\ &= \exp\left(i\sum_{j=1}^N a_j \left(-(r_2^j - r_0^j)^\mu + (r_2^j - r_1^j)^\mu + (r_1^j - r_0^j)^\mu\right)\right).\end{aligned}$$

It is easy to prove the following local analogue of Proposition 3.13.

Lemma 3.17. *For $[\mathscr{A}_d^s] \in S^{1,\mathscr{V}''}(M,G)$ and $[\mathscr{B}_d^s] \in S^{2,\mathscr{U}''}(M,G)$ as above, we have $[\mathscr{B}_d^s] = \delta_1^{\mathscr{V}''}[\mathscr{A}_d^s]$ and $\delta_2^{\mathscr{V}''}[\mathscr{B}_d^s] = e$.*

4. Holonomy around a loop

In this section we find an explicit formula for the discrete holonomy phase around a loop that is contained in an open set trivializing π. We still work in the case where G, the structure group of π, is abelian.

Remark 4.1. By Proposition 2.17 the different values of the discrete holonomy phase around a loop, starting at different points of the corresponding fiber are conjugated elements of G. When G is abelian, these elements are all the same, so that the discrete phase around a loop only depends on the loop and we denote it by $\Phi_d(m_\cdot)$.

Lemma 4.2. *Let $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π, whose structure group G is abelian. Let $V \subset M$ be a connected open subset and $s : M \to Q$ be a local section of π. For any discrete loop in M, $m_\cdot \in \Omega_{N,\mathscr{V}''}(\overline{m})$, we have that*

(1) *there exists* $\widetilde{m} = \sum_{k=1}^{N} T_1^k \in S_{1,\mathscr{V}''}(M)$ *such that* $T_1^k(e_0) = m_{k-1}$ *and* $T_1^k(e_1) = m_k$ *for* $k = 1, \ldots, N$.

(2) *Then, for any such* $\widetilde{m}$*,* $\prod_{k=1}^{N} \mathscr{A}_d^s(m_{k-1}, m_k)^{-1} = \left(\int_{\widetilde{m}} [\mathscr{A}_d^s]\right)^{-1}$.

Proof. Being V connected, for each $k = 1, \ldots, N$ there is a continuous path $\gamma_{k-1,k} : [0,1] \to V$ such that $\gamma_{k-1,k}(0) = m_{k-1}$ and $\gamma_{k-1,k}(1) = m_k$. Define $T_1^k : \Delta_1 \to M$ by $T_1^k((1-t)e_0 + te_1) := \gamma_{k-1,k}(t)$ for $t \in [0,1]$. As $(T_1^k(e_i), T_1^k(e_j)) \in \mathscr{V}''$ for all $i, j = 0, 1$, we see that $\widetilde{m} := \sum_k^N T_1^k \in S_{1,\mathscr{V}''}(M)$, proving point (1). Then, by definition of $\int$,

$$\begin{aligned}\int_{\widetilde{m}} [\mathscr{A}_d^s] = \mathscr{A}_d^s(\widetilde{m}) = \mathscr{A}_d^s\left(\sum_{k=1}^{N} T_1^k\right) = \prod_{k=1}^{N} \mathscr{A}_d^s(T_1^k) \\ = \prod_{k=1}^{N} \mathscr{A}_d^s(T_1^k(e_0), T_1^k(e_1)) = \prod_{k=1}^{N} \mathscr{A}_d^s(m_{k-1}, m_k),\end{aligned}$$

that, on inversion, leads to point (2). □

We say that the chain $\widetilde{m} \in S_{1,\mathscr{V}''}(M)$ satisfying point (1) of Lemma 4.2 *interpolates* the discrete path $m_{\cdot}$.

Theorem 4.3. *Let* $\mathscr{A}_d : \mathscr{U} \to G$ *be a discrete connection on the principal G-bundle* $\pi : Q \to M$ *with G abelian,* $V \subset M$ *a connected open subset and* $s : V \to Q$ *a local section of* π*. For any* $m_{\cdot} \in \Omega_N(\overline{m})$ *such that* $(m_0, m_k, m_{k+1}) \in \mathscr{V}''^{(3)}$ *for all* $k = 0, \ldots, N-1$*, we have*

$$\Phi_d(m_{\cdot}) = \left(\int_{\widetilde{m}} [\mathscr{A}_d^s]\right)^{-1}, \tag{4.1}$$

where $\widetilde{m} \in S_{1,\mathscr{V}''}(M)$ *is any small singular chain interpolating the discrete path* $m_{\cdot}$*, in the sense of Lemma 4.2. If, in addition, there is* $\widetilde{\sigma} \in S_{2,\mathscr{V}''}(M)$ *so that* $\partial_2^{\mathscr{V}''}(\widetilde{\sigma}) = \widetilde{m}$*, then*

$$\Phi_d(m_{\cdot}) = \left(\int_{\widetilde{\sigma}} [\mathscr{B}_d^s]\right)^{-1}. \tag{4.2}$$

Proof. Identity (4.1) follows from the first equality in (2.8) taking into account that G is abelian and, then, point (2) of Lemma 4.2. On the other hand, if $\widetilde{\sigma} \in S_{2,\mathscr{V}''}(M)$ satisfies $\partial_2^{\mathscr{V}''}(\widetilde{\sigma}) = \widetilde{m}$, using Proposition 3.17 and (a “small version” of) (3.1), we have

$$\int_{\widetilde{\sigma}} [\mathscr{B}_d^s] = \int_{\widetilde{\sigma}} \delta_1^{\mathscr{V}''}[\mathscr{A}_d^s] = \int_{\partial_2^{\mathscr{V}''}\widetilde{\sigma}} [\mathscr{A}_d^s] = \int_{\widetilde{m}} [\mathscr{A}_d^s],$$

and (4.2) follows from (4.1). □

Example 4.4. In the case of the discrete connection form $\mathscr{A}_{d,\mu}$ introduced in Example 2.5 together with the (global) trivialization of Example 2.11, if $r_{\cdot} \in \Omega_{N,\mathscr{V}''}(\overline{r}) = \Omega_N(\overline{r})$, it automatically satisfies that $(r_0, r_k, r_{k+1}) \in \mathscr{V}''^{(3)} = \mathbb{R}^3_{>0}$ for all k, so that we can apply Theorem 4.3 to compute the discrete holonomy phase $\Phi_d(r_{\cdot})$. A 1-chain in $\mathbb{R}_{>0}$ that interpolates $r_{\cdot}$ is $\widetilde{r} := \sum_{j=1}^{N} T_1^j$ for $T_1^j(t_0e_0 + t_1e_1) := t_0r_{j-1} + t_1r_j$, so that $\widetilde{r} \in S_1(\mathbb{R}_{>0})$. Thus, using (4.1) and Example 3.16, we have

$$\Phi_d(r_{\cdot}) = \left(\int_{\widetilde{r}} [\mathscr{A}_{d,\mu}]\right)^{-1} = ([\mathscr{A}_{d,\mu}](\widetilde{r}))^{-1} = \exp\left(-i\sum_{j=1}^{N}(r_j - r_{j-1})^{\mu}\right). \tag{4.3}$$

We construct $\widetilde{\sigma} \in S_2(\mathbb{R}_{>0})$ as follows. First, for each $j = 1, \dots, N-1$, define

$$\gamma_j(t) := \begin{cases} 3tr_j + (1-3t)r_0, & \text{if } 0 \leq t \leq \frac{1}{3}, \\ (3t-1)r_{j+1} + (2-3t)r_j, & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (3t-2)r_0 + (3-3t)r_{j+1}, & \text{if } \frac{2}{3} \leq t \leq 1, \end{cases}$$

and $T_2^j(t_0e_0 + t_1e_1 + t_2e_2) := \gamma_j(\frac{1}{3}t_1 + \frac{2}{3}t_2)$. Finally, $\widetilde{\sigma} := \sum_{j=1}^{N-1} T_2^j$. By construction, $\widetilde{\sigma} \in S_2(\mathbb{R}_{>0})$ and it is not hard to check that $\partial_2\widetilde{\sigma} = \widetilde{r}$. Then, using (4.2) and (2.7),

$$\begin{aligned}
\Phi_d(m_.) &= \left(\int_{\widetilde{\sigma}} [\mathscr{B}^s_{d,\mu}]\right)^{-1} = \left([\mathscr{B}^s_{d,\mu}](\widetilde{\sigma})\right)^{-1} \\
&= \prod_{j=1}^{N-1} (\mathscr{B}^s_{d,\mu}(T_2^j(1,0,0), T_2^j(0,1,0), T_2^j(0,0,1)))^{-1} = \prod_{j=1}^{N-1} (\mathscr{B}^s_{d,\mu}(r_0, r_j, r_{j+1}))^{-1} \\
&= \prod_{j=1}^{N-1} \exp(-i(-(r_{j+1}-r_0)^\mu + (r_{j+1}-r_j)^\mu + (r_j - r_0)^\mu)) \\
&= \exp\left(-i\sum_{j=1}^{N-1}(-(r_{j+1}-r_0)^\mu + (r_{j+1}-r_j)^\mu + (r_j-r_0)^\mu)\right) \\
&= \exp\left(-i\sum_{j=0}^{N-1}(r_{j+1}-r_j)^\mu\right),
\end{aligned}$$

matching the previous computation of $\Phi_d(m_.)$.

Remark 4.5. For (continuous) connections on a principal G-bundle the set of all possible holonomy phases starting at a given point is known as the holonomy group of the connection and it encodes interesting information about the connection and the bundle (see, for instance, [KN96]). A similar set can be constructed in the discrete setting: $\text{Hol}_{\mathscr{A}_d}(\overline{q}) := \{\Phi_d(m_., \overline{q}) \in G : m_. \in \Omega_{\mathscr{U}''}(\pi(\overline{q}))\}$. As a subset of the group G, $\text{Hol}_{\mathscr{A}_d}(\overline{q})$ is, in general, a submonoid (*i.e.*, it contains the identity element and is closed under products). If, in addition, the discrete connection is *symmetric* (essentially, $\mathscr{A}_d(q_0,q_1) = \mathscr{A}_d(q_1,q_0)^{-1}$ for all $(q_0,q_1) \in \mathscr{U}$), $\text{Hol}_{\mathscr{A}_d}(\overline{q})$ is a subgroup of G (see [FZ] for more details on discrete holonomy).

For the discrete connection $\mathscr{A}_{d,\mu}$ introduced in Example 2.5, it is easy to see, using (4.3), that $\text{Hol}_{\mathscr{A}_d}(\overline{q}) = U(1)$ for all $\mu > 1$ while $\text{Hol}_{\mathscr{A}_d}(\overline{q}) = \{1\}$ if $\mu = 1$.

Notice that identities (4.1) and (4.2) are discrete analogues of (1.1) and (1.2), although not "exponentiated". In the next section we make a few formal changes to consider singular cochains with values in $\mathfrak{g} := Lie(G)$, so that we can obtain exact discrete analogues of (1.1) and (1.2).

5. Forms with values in the Lie algebra

The goal of this section is to define functions a^s_d and b^s_d with values in $\mathfrak{g}$ so that $\mathscr{A}^s_d = \exp_G(a^s_d)$ and $\mathscr{B}^s_d = \exp_G(b^s_d)$. Then, use them to define singular cochains that will enable us to prove discrete analogues of (1.1) and (1.2).

5.1. **Logarithms of $\mathscr{A}^s_d$ and $\mathscr{B}^s_d$.** The main tool that we need is the following classical result.

Theorem 5.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, the exponential map $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism between open neighborhoods U_0 and V_e of $0 \in \mathfrak{g}$ and $e \in G$. In addition, when G is abelian, $\exp$ is a homomorphism of groups, considering $\mathfrak{g}$ as a Lie group with the operation given by the addition.*

Proof. See Theorem 2.10.1 and Corollary 2.13.3 of [Var84]. □

The following result uses Theorem 5.1 to obtain some additional open sets.

Lemma 5.2. *With the same notation of Theorem 5.1, there are open neighborhoods $U_0' \subset U_0$ and $V_e' \subset V_e$ of $0 \in \mathfrak{g}$ and $e \in G$ such that $\exp_G|_{U_0'} : U_0' \to V_e'$ is a diffeomorphism and the following conditions hold.*

1. *U_0' and V_e' are invariant under $a \mapsto -a$ and $g \mapsto g^{-1}$ respectively.*
2. *For any $a_0, a_1, a_2 \in U_0'$, $a_0 + a_1 + a_2 \in U_0$ and, when G is abelian, for any $g_0, g_1, g_2 \in V_e'$, we have $g_0 g_1 g_2 \in V_e$.*

Proof. It is a standard application of topological properties, mostly continuity of the group and algebra operations. □

In what follows we assume that the structure group G of π is abelian and that $\mathscr{A}_d : \mathscr{U} \to G$ is a discrete connection form on π. Also, we fix an open subset $V \subset M$ and a section $s : V \to Q$ of π.

Definition 5.3. Let $\mathscr{W}'' := (\mathscr{A}_d^s)^{-1}(V_e') \subset M \times M$ and $a_d^s : \mathscr{W}'' \to U_0'$ so that $\exp_G \circ a_d^s = \mathscr{A}_d^s$. We call a_d^s the *logarithm of the local expression of $\mathscr{A}_d$*.

Lemma 5.4. *In the context of Definition 5.3, $\mathscr{W}'' \subset \mathscr{V}''$ is open, and a_d^s is well defined and smooth.*

Proof. As $\mathscr{A}_d^s : \mathscr{V}'' \to G$ is smooth, and $V_e' \subset G$ is open, $\mathscr{W}'' \subset \mathscr{V}''$ is open. For any $(m_0, m_1) \in \mathscr{W}''$, we have $\mathscr{A}_d^s(m_0, m_1) \in V_e'$ and, as $\exp_G|_{U_0'} : U_0' \to V_e'$ is a diffeomorphism, there is a unique $a_d^s(m_0, m_1) \in U_0'$ such that $\exp_G(a_d^s(m_0, m_1)) = \mathscr{A}_d^s(m_0, m_1)$. The smoothness of a_d^s follows from that of $\mathscr{A}_d^s$ and the fact that $\exp_G|_{U_0'}$ is a diffeomorphism. □

In the same spirit, we have the following notion.

Definition 5.5. Let $\widetilde{\mathscr{W}''} := (\mathscr{B}_d^s)^{-1}(V_e) \subset M^3$ and $b_d^s : \widetilde{\mathscr{W}''} \to U_0$ so that $\exp_G \circ b_d^s = \mathscr{B}_d^s$. We call b_d^s the *logarithm of the local expression of $\mathscr{B}_d$*.

Lemma 5.6. *In the context of Definition 5.5, we have*

1. *$\widetilde{\mathscr{W}''} \subset \mathscr{V}''^{(3)}$ is open and $\mathscr{W}''^{(3)} \subset \widetilde{\mathscr{W}''}$, where $\mathscr{V}''$ is introduced in Definition 2.10, $\mathscr{W}''$ in Definition 5.3 and $\mathscr{V}''^{(3)}, \mathscr{W}''^{(3)}$ follow (3.2).*
2. *b_d^s is well defined and smooth.*
3. *For all $(m_0, m_1, m_2) \in \mathscr{W}''^{(3)}$, we have $b_d^s(m_0, m_1, m_2) = -a_d^s(m_0, m_2) + a_d^s(m_1, m_2) + a_d^s(m_0, m_1)$.*

Proof. As $\mathscr{B}_d^s : \mathscr{V}''^{(3)} \to G$, $\widetilde{\mathscr{W}''} = \mathscr{B}_d^{s\,-1}(V_e') \subset \mathscr{V}''^{(3)}$ and its openness follows from that of V_e' and the continuity of $\mathscr{B}_d^s$. On the other hand, for $(m_0, m_1, m_2) \in \mathscr{W}''^{(3)}$, we have $(m_j, m_k) \in \mathscr{W}''$ for all $j, k = 0, 1, 2$ and, so, $\mathscr{A}_d^s(m_j, m_k) \in V_e'$ for all $j, k = 0, 1, 2$. Then, by Lemma 2.15 and point (2) of Lemma 5.2, we have

$$\mathscr{B}_d^s(m_0, m_1, m_2) = \mathscr{A}_d^s(m_0, m_2)^{-1} \mathscr{A}_d^s(m_1, m_2) \mathscr{A}_d^s(m_0, m_1) \in V_e.$$

Thus, $(m_0,m_1,m_2)\in(\mathscr{B}_d^s)^{-1}(V_e)$, proving point (1). As $\mathscr{B}_d^s(\widetilde{\mathscr{W}''})\subset V_e$ and $\exp_G|_{U_0}:U_0\to V_e$ is a diffeomorphism, $b_d^s=(\exp_G|_{U_0})^{-1}\circ\mathscr{B}_d^s|_{\widetilde{\mathscr{W}''}}$ is well defined and smooth, proving point (2). In order to prove point (3) we observe that, for $(m_0,m_1,m_2)\in\mathscr{W}''^{(3)}$,

$$\begin{aligned}\exp_G(b_d^s(m_0,m_1,m_2)) &= \mathscr{B}_d^s(m_0,m_1,m_2)=\mathscr{A}_d^s(m_0,m_2)^{-1}\mathscr{A}_d^s(m_1,m_2)\mathscr{A}_d^s(m_0,m_1)\\ &=(\exp_G(a_d^s(m_0,m_2)))^{-1}\exp_G(a_d^s(m_1,m_2))\exp_G(a_d^s(m_0,m_1)) \quad (5.1)\\ &=\exp_G(-a_d^s(m_0,m_2)+a_d^s(m_1,m_2)+a_d^s(m_0,m_1)),\end{aligned}$$

where the second equality is by Lemma 2.15, the third is because $(m_j,m_k)\in\mathscr{W}''$ for all j,k and the last by Theorem 5.1 and the commutativity of G. Then, as $a_d^s(m_j,m_k)\in U_0'$, by Lemma 5.2, $-a_d^s(m_0,m_2)+a_d^s(m_1,m_2)+a_d^s(m_0,m_1)\in U_0$. As we also have that $b_d^s(m_0,m_1,m_2)\in U_0$ and we know that $\exp_G$ is injective over U_0, point (3) of the statement now follows from (5.1). □

Example 5.7. In the context of Example 2.5, we have that

$$\mathfrak{g}:=Lie(U(1))=i\mathbb{R} \quad\text{and}\quad \exp_{U(1)}(i\zeta)=\exp(i\zeta).$$

If we take $U_0:=i(-\pi,\pi)$ and $V_e:=U(1)\setminus\{-1\}$, it is an elementary fact that $\exp_{U(1)}|_{U_0}:U_0\to V_e$ is a diffeomorphism; the corresponding inverse map is $\mathrm{Log}(z)$, (the appropriate restriction of) the principal branch of the complex logarithm. The sets U_0' and V_e' constructed in the proof of Lemma 5.2 are $U_0'=i(-\frac{\pi}{3},\frac{\pi}{3})$ and $V_e'=\{z\in U(1):|\mathrm{Arg}(z)|\leq\frac{\pi}{3}\}$ (where $\mathrm{Arg}(z)$ is the argument of z that lies in $(-\pi,\pi]$). Following Definition 5.3, the domain of $a_{d,\mu}^s$ is $\mathscr{W}''=(\mathscr{A}_d^s)^{-1}(V_e')=\{(r_0,r_1)\in\mathbb{R}^2_{>0}:|\mathrm{mod}_{2\pi}((r_1-r_0)^\mu)|<\frac{\pi}{3}\}$, where $\mathrm{mod}_{2\pi}(r)$ is the unique real number in $(-\pi,\pi]$ congruent to r modulo 2π; then, for $(r_0,r_1)\in\mathscr{W}''$,

$$a_{d,\mu}^s(r_0,r_1)=\mathrm{Log}(\mathscr{A}_{d,\mu}^s(r_0,r_1))=\mathrm{mod}_{2\pi}((r_1-r_0)^\mu).$$

Similarly, following Definition 5.5, we have $\widetilde{\mathscr{W}''}:=(\mathscr{B}_d^s)^{-1}(V_e)=\{(r_0,r_1,r_2)\in\mathbb{R}^3_{>0}:\mathrm{mod}_{2\pi}(-(r_2-r_0)^\mu+(r_1-r_0)^\mu+(r_2-r_0)^\mu)\neq\pi\}$ and, for all $(r_0,r_1,r_2)\in\widetilde{\mathscr{W}''}$,

$$b_{d,\mu}^s(r_0,r_1,r_2)=\mathrm{Log}(\mathscr{B}_d^s(r_0,r_1,r_2))=\mathrm{mod}_{2\pi}(-(r_2-r_0)^\mu+(r_1-r_0)^\mu+(r_2-r_0)^\mu).$$

5.2. **Small singular cochains associated to a_d^s and b_d^s.** Now we reproduce the arguments of Section 3.2 to obtain small singular cochains with values in $\mathfrak{g}$ associated to a_d^s and b_d^s.

Recall the open subset $\mathscr{W}''\subset M\times M$ introduced in Definition 5.3. In what follows, we work with the $\mathscr{W}''$-small singular cochain complex $(S_{n,\mathscr{W}''}(M,\mathfrak{g}),\partial_n^{\mathscr{W}''})$.

Definition 5.8. With the notation as above, for any 1-simplex $T_1\in S_{1,\mathscr{W}''}(M,\mathfrak{g})$ we define

$$[a_d^s](T_1):=a_d^s(T_1(1,0),T_1(0,1)).$$

Also, for any 2-simplex $T_2\in S_{2,\mathscr{W}''}(M,\mathfrak{g})$ we define

$$[b_d^s](T_2):=b_d^s(T_2(1,0,0),T_2(0,1,0),T_2(0,0,1)).$$

As the 1 and 2 simplexes freely generate $S_{1,\mathscr{W}''}(M)$ and $S_{2,\mathscr{W}''}(M)$ respectively, the previous formulas uniquely define $[a_d^s]\in S^{1,\mathscr{W}''}(M,\mathfrak{g})$ and $[b_d^s]\in S^{2,\mathscr{W}''}(M,\mathfrak{g})$.

Proposition 5.9. *With the same notation as above, $\delta_1^{\mathscr{W}''}[a_d^s]=[b_d^s]$ and $\delta_2^{\mathscr{W}''}[b_d^s]=0$.*

Proof. As the 2-simplexes generate $S_{2,\mathscr{W}''}(M)$, it suffices to verify that $(\delta_1^{\mathscr{W}''}[a_d^s])(T_2)=[b_d^s](T_2)$ for every such 2-simplex T_2. After unraveling the definitions, this identity follows from point (3) of Lemma 5.6 and (a "small version" of) the formula in Example 3.4. Thus, the first identity holds. The second one is a consequence of the first and of Lemma 3.11. □

Proposition 5.10. *The cochains $[\mathscr{A}_d^s] \in S^{1,\mathscr{V}''}(X,G)$ and $[\mathscr{B}_d^s] \in S^{2,\mathscr{V}''}(X,G)$ are naturally elements of $S^{1,\mathscr{W}''}(X,G)$ and $S^{2,\mathscr{W}''}(X,G)$ respectively. As such, we have*

$$[\mathscr{A}_d^s] = (\exp_G)_*[a_d^s] \quad \text{and} \quad [\mathscr{B}_d^s] = (\exp_G)_*[b_d^s],$$

where $(\exp_G)_*$ is the homomorphism of cochain complexes defined in Section 3.1 and induced by $\exp_G \in \hom(\mathfrak{g},G)$.

Proof. That the $\mathscr{V}''$-small cochains are also $\mathscr{W}''$-small cochains follows from $\mathscr{W}'' \subset \mathscr{V}''$ (Lemma 5.4) and, then, because $S_{n,\mathscr{W}''}(M) \subset S_{n,\mathscr{V}''}(M)$. To check the identities, it suffices to see that they are satisfied on 1 and 2 simplexes $T_1 \in S_{1,\mathscr{W}''}(M)$ and $T_2 \in S_{2,\mathscr{W}''}(M)$. But then, on evaluation both identities are satisfied because of Definitions 5.3 and 5.5. □

Theorem 5.11. *Let $\pi : Q \to M$ be a principal G-bundle with G abelian and $\mathscr{A}_d : \mathscr{U} \to G$ be a discrete connection on π. Given an open subset $V \subset M$ and $s : V \to Q$ a smooth section of π as well as $\overline{m} \in V$, for any $m_. \in \Omega_N(\overline{m})$ such that $(m_0, m_k, m_{k+1}) \in \mathscr{W}''^{(3)}$ for all $k = 0, \ldots, N-1$, we have that the discrete holonomy phase around $m_.$ is*

$$\Phi_d(m_.) = \exp_G\left(-\int_{\widetilde{m}} [a_d^s]\right), \tag{5.2}$$

where $\widetilde{m} \in S_{1,\mathscr{W}''}(M)$ interpolates $m_.$, that is, satisfies the conditions of Lemma 4.2. If, in addition, there is $\widetilde{\sigma} \in S_{2,\mathscr{W}''}(M)$ such that $\partial_2^{\mathscr{W}''}(\widetilde{\sigma}) = \widetilde{m}$, then

$$\Phi_d(m_.) = \exp_G\left(-\int_{\widetilde{\sigma}} [b_d^s]\right). \tag{5.3}$$

Proof. Observe that $\widetilde{m} \in S_{1,\mathscr{W}''}(M) \subset S_{1,\mathscr{V}''}(M)$ so that, by Theorem 4.3, we have (4.1). Then, using Proposition 5.10 and the fact that $\exp_G$ is a homomorphism,

$$\begin{aligned}\Phi_d(m_.) &= \left(\int_{\widetilde{m}} [\mathscr{A}_d^s]\right)^{-1} = ([\mathscr{A}_d^s](\widetilde{m}))^{-1} = (((\exp_G)_*[a_d^s])(\widetilde{m}))^{-1} = \exp_G([a_d^s](\widetilde{m}))^{-1} \\ &= \exp_G(-[a_d^s](\widetilde{m})) = \exp_G\left(-\int_{\widetilde{m}} [a_d^s]\right),\end{aligned}$$

proving (5.2). Then, if $\partial_2^{\mathscr{W}''}(\widetilde{\sigma}) = \widetilde{m}$, (5.3) follows immediately from (5.2) and

$$\int_{\widetilde{m}} [a_d^s] = \int_{\partial_2^{\mathscr{W}''}(\widetilde{\sigma})} [a_d^s] = \int_{\widetilde{\sigma}} \delta_1^{\mathscr{W}''}[a_d^s] = \int_{\widetilde{\sigma}} [b_d^s].$$

□

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