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ON BEST LOCAL APPROXIMATION IN L^p SPACES

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ABSTRACT. The purpose of this paper is to revise certain aspects concerning the best local approximation of functions. We shall consider the problem of best local simultaneous approximation to two functions at a single point, and the problem of best local approximation to a function on several points assuming weaker conditions than usual.

1. INTRODUCTION

The problem of best local approximation was introduced and studied in a paper by Chui, Smith and Shisha [5]. The initiation of this could be dated back to results of J. L. Walsh, who proved that the Padé approximant of an analytic function over a domain (real or complex) can be obtained as a "limit" of the net of the best rational approximants by shrinking the domain to a single point. A special case of this, when approximating functions are chosen to be polynomials of certain degree, the limit is the Taylor polynomial of the same degree which was already known to Walsh in 1934 [19].

The results mentioned above concerned the problem over a single domain (real or complex). In them was studied the limiting behavior of certain approximating functions when the domain shrinks to a single point.

In 1979, Su [18] studied the problem in two disjoint intervals. This study sets up the initial steps for multipoint best local approximation.

Later, Beatson and Chui [2] formally introduced the multipoint problem and they obtained some results in L^p spaces.

In 1986, Marano [16] proved the existence of the best local approximant in L^p spaces assuming certain order of differentiability in the L^p sense.

In 2003, the authors [12] used a technique similar to that of the last work, and they proved existence of the best local approximant, assuming lateral differentiability in the L^p sense. Also, they established a necessary condition in L^2 spaces.

In [10] was studied the best simultaneous local approximation problem for the norms $l_\infty - L^p$ in the case of one point and in the case of several points when $p = 2$. Later, in [9] it was considered this problem for the norms $l_q - L^p$ and it was obtained interpolation results of best simultaneous approximants.

In this paper we will introduce the mentioned results in three sections after defining best local approximant and the concept of lateral differentiability in the L^p sense. We shall not give here the proof of the different theorems, we only indicate the tools that we have used in previous works, and the lemmas that we have proved in order to establish the main results.

Finally, we observe that the multipoint best local approximation problem also was researched in the consideration of more general norms that L^p norms. For example, it was studied for Orlicz norms and for more general families of neighborhoods (see [6, 8, 11, 13]). However, in this article we will only consider L^p norms and the neighborhoods will be a union of intervals centered on a finite set of points with the same measure.

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Let $x_1, x_2, \dots, x_k$ be k points in $\mathbb{R}$ and let $a > 0$ be such that the intervals $[x_i - a, x_i + a]$, $1 \le i \le k$, are pairwise disjoint. Let $\mathscr{L}$ be the space of real Lebesgue measurable functions defined on $A_a := \bigcup_{i=1}^{k} [x_i - a, x_i + a]$. For each Lebesgue measurable set $A \subset A_a$, with $|A| > 0$, we consider the semi-norms on $\mathscr{L}$,

$$\|h\|_{p,A} := \left(\frac{1}{|A|}\int_A |h(x)|^p dx\right)^{1/p}, \ 1 \le p < \infty, \qquad \text{and} \qquad \|h\|_{\infty,A} := \sup_{x\in A}\text{ess}\,|h(x)|,$$

where $|A|$ denotes the measure of the set A.

If $0 < \varepsilon \le a$, $A_{-\varepsilon,i} := [x_i - \varepsilon, x_i]$, $A_{+\varepsilon,i} := [x_i, x_i + \varepsilon]$, and $A_{\varepsilon,i} := [x_i - \varepsilon, x_i + \varepsilon]$, we write

$$\begin{aligned} \|h\|_{p,-\varepsilon,i} &= \|h\|_{p,A_{-\varepsilon,i}}, \\ \|h\|_{p,+\varepsilon,i} &= \|h\|_{p,A_{+\varepsilon,i}}, \\ \|h\|_{p,\varepsilon,i} &= \|h\|_{p,A_{\varepsilon,i}}, \\ \|h\|_{p,\varepsilon} &= (\sum_{i=1}^{k} \|h\|_{p,\varepsilon,i}^p)^{1/p}. \end{aligned}$$

For $s \in \mathbb{N}$, we denote Π^s the set of polynomials of degree at most s.

The next concept was introduced by Calderón and Zygmund in [3]. We say that h is *right differentiable (left differentiable) in the L^p sense at x_i up to order m* if there is a polynomial $R_i \in \Pi^m (L_i \in \Pi^m)$ such that

$$\|h - R_i\|_{p,+\varepsilon,i} = o(\varepsilon^m) \quad (\|h - L_i\|_{p,-\varepsilon,i} = o(\varepsilon^m)). \tag{1}$$

In addition, if $R_i = L_i$ we say that h is *differentiable in the L^p sense at x_i up to order m*. It is easy to prove that if h is right (left) differentiable up to order m at x_i then R_i (L_i) is unique. Furthermore, h is right differentiable up to order $m-1$ at x_i, and if

$$R_i(x) = S_i(x) + b_i x^m, \quad S_i \in \Pi^{m-1},$$

then S_i is the unique polynomial in Π^{m-1} which verifies $\|h - S_i\|_{p,+\varepsilon,i} = o(\varepsilon^{m-1})$. So, we can define the right derivatives of f in the L^p sense at x_i of order j, $0 \le j \le m$, by the corresponding derivatives of R_i, i.e.,

$$f_+^{(j)}(x_i) := R_i^{(j)}(x_i).$$

Analogously we define the left derivatives. If $f_+^{(j)}(x_i) = f_-^{(j)}(x_i)$, we denote it by $f^{(j)}(x_i)$ and it is called the derivative in the L^p sense of f of order j at x_i. As usual, we can define the right and left Taylor polynomials, which are denoted by $T_{+,i}^m(h)$ and $T_{-,i}^m(h)$, respectively. If $T_{+,i}^m(h) = T_{+,i}^m(h)$, we denote it by $T_i^m(h)$.

If $h \in L^p(A)$, $1 \le p \le \infty$, and $A \subset A_a$, it is well known [4] that there is a *best approximant to h from Π^n respect to the semi-norm $\|\cdot\|_{p,A}$*, say $P_A(h)$; i.e., $P_A(h)$ verifies

$$\|h - P_A(h)\|_{p,A} \le \|h - Q\|_{p,A}, \quad \text{for all } Q \in \Pi^n.$$

When $1 < p < \infty$ such a polynomial is unique. For $A = A_\varepsilon := \cup_{i=1}^k A_{\varepsilon,i}$ we write $P_A(h) = P_\varepsilon(h)$. If there exists the $\lim_{\varepsilon\to 0} P_\varepsilon(h)$, it is called *the best local approximant of h on the set* $\{x_i : 1 \le i \le k\}$, and it will be denoted by $P_0(h)$.

We also need to introduce the concept of best simultaneous local approximant to two functions $h_1, h_2 \in L^p(A)$. Given a norm in $\mathbb{R}^2$, say ρ, we consider a $\rho - L^p$ *best simultaneous* *approximant to* h_1 *and* h_2 *from* Π^n, say $P_\varepsilon(h_1,h_2)$, respect to the norms ρ and L^p, i.e., $P_\varepsilon(h_1,h_2)$ verifies

$$\rho(\|h_1-P_\varepsilon(h_1,h_2)\|_{p,\varepsilon},\|h_2-P_\varepsilon(h_1,h_2)\|_{p,\varepsilon}) \le \rho(\|h_1-Q\|_{p,\varepsilon}\|h_2-Q\|_{p,\varepsilon}), \quad \text{for all } Q\in\Pi^n.$$

It is known that if $1<p<\infty$ and ρ is the l_q-norm with $1<q<\infty$, then there exists a unique l_q-L^p best simultaneous approximant. In fact, it is a consequence of $(\|h_1\|_p^q+\|h_2\|_p^q)^{1/q}$ being a strictly convex norm.

2. Lateral differentiability and best local approximation

As usual, $\mathscr{C}^m(A)$ denotes the set of functions continuously differentiable on A up to order m.

The following theorem generalizes the one point L^p result in Su [18] to k points, where $k\ge 2$, by Beatson and Chui [2].

Theorem 1. *Let* $1<p<\infty$ *and* $n+1=kq$. *For each* $f\in\mathscr{C}^{q-1}$, *the net of best approximants* $\{P_\varepsilon(f)\}$ *converges to some polynomial* $P_0\in\Pi^n$. *Furthemore,* P_0 *is the unique polynomial in* Π^n *which satisfies the interpolation conditions*

$$P_0^{(j)}(x_i)=f^{(j)}(x_i), \quad 1\le i\le k,\ 0\le j\le q-1.$$

Clearly, in this case the number of parameters match with the number of "data" at $x_i, 1\le i\le k$, and the best local approximant turned out to be the Hermite interpolation polynomial which enjoys the oscillating interpolating property evenly over the k points.

What happens if the number of parameters does not match the number of data?

In [CDR] the authors extended Theorem 1 when $n+1=kq+r$, $r\ge 0$, and f is a function in $\mathscr{C}^q$.

Later, in [16] with a different technique, the author also extended Theorem 1 when $n+1=kq+r$, $r\ge 0$, and f is a function differentiable in the L^p sense up to order q at x_i, $1\le i\le k$. More precisely, he proved,

Theorem 2. *Let* $1<p\le\infty$ *and* $n+1=kq+r$, $r\ge 0$. *Let* $f\in\mathscr{L}$ *be differentiable in the* L^p *sense up to order* q *at* x_i, $1\le i\le k$, *if* $r>0$ *(up to order* $q-1$ *if* $r=0$*). If* $r>0$, *the net of best approximants* $\{P_\varepsilon(f)\}$ *converges to a polynomial* P_0 *which is the unique polynomial in* Π^n *that minimizes*

$$\sum_{i=1}^{k}|f^{(q)}(x_i)-Q^{(q)}(x_i)|^p, \quad 1<p<\infty \qquad \text{or} \qquad \max_{1\le i\le k}|f^{(q)}(x_i)-Q^{(q)}(x_i)|, \quad p=\infty,$$

and verifies the condition

$$Q^{(j)}(x_i)=f^{(j)}(x_i), \quad 1\le i\le k,\ 0\le j\le q-1. \tag{2}$$

If $r=0$ *the net of best approximants* $\{P_\varepsilon(f)\}$ *converges to a polynomial* P_0 *which is the unique polynomial in* Π^n *that verifies (2).*

Using a technique similar to that employed in [16], the authors proved in [12] the following extension of Theorem 2 in the multiple case, i.e., $r=0$.

Theorem 3. *Let* $1<p\le\infty$ *and* $n+1=kq$. *Let* f *be a function with derivatives in the* L^p *sense up to order* $q-2$ *at* x_i, *and with lateral derivatives in the sense* L^p *of order* $q-1$ *at* x_i, $1\le i\le k$. *Then the best local approximant of* f *exists, and it is equal to*

$$\frac{H_-(f)+H_+(f)}{2},$$

where $H_-(f)$ and $H_+(f)$ are the polynomials in Π^n univocally determined by the conditions $H_-^{(j)}(x_i) = f_-^{(j)}(x_i)$, $H_+^{(j)}(x_i) = f_+^{(j)}(x_i)$, $1 \le i \le k$, $0 \le j \le q-1$.

We introduce an additional notation and establish some results which were necessary to prove Theorem 3 in [12]. However, as already mentioned in the Introduction, this will not be shown here.

Assume that f has left and right derivatives in the L^p sense at $x_i, 1 \le i \le k$, up to order $q-1$, and $T_{-,i}^{q-2}(f) = T_{+,i}^{q-2}(f)$. We consider the following set

$$\mathscr{H}(f) = \{Q \in \Pi^n : Q^{(j)}(x_i) = f^{(j)}(x_i),\ 0 \le j \le q-2,\ 1 \le i \le k\}.$$

If $q = 1$ we put $\mathscr{H}(f) = \{0\}$. Fix $H_0 \in \mathscr{H}(f)$ and consider the function $g = f - H_0$. Clearly, $g^{(j)}(x_i) = 0$, $1 \le j \le q-2, 1 \le i \le k$. The next lemma immediately follows.

Lemma 1. *Let $P \in \Pi^n$, $1 < p \le \infty$, and $0 < \varepsilon \le a$. Then P is a best $\|\cdot\|_{p,\varepsilon}$-approximant to f from Π^n if and only if $P = H_0 + Q$, with Q a best $\|\cdot\|_{p,\varepsilon}$-approximant to g from Π^n. In addition, the best local approximant to f exists if and only if the best local approximant to g exists, and $P_0(f) = P_0(g) + H_0$.*

By Lemma 1 it is sufficient for our purposes to prove the existence and characterization of $P_0(g)$. The following lemma assures that the set of accumulation points of the net $\{P_\varepsilon(g)\}$ is not empty.

Lemma 2. *Let $1 < p \le \infty$. The set of polynomials $\{P_\varepsilon(g)\}$, for ε small, is uniformly bounded on compact sets.*

If $P \in \Pi^n$, $u_i = \frac{g_-^{(q-1)}(x_i)}{(q-1)!}$ and $v_i = \frac{g_+^{(q-1)}(x_i)}{(q-1)!}$, we consider the following function defined on A_a,

$$P^*(x) = \begin{cases} u_i(x-x_i)^{q-1} - T_i^{q-1}(P)(x), & \text{if } x \in [x_i - a, x_i], \\ v_i(x-x_i)^{q-1} - T_i^{q-1}(P)(x), & \text{if } x \in (x_i, x_i + a]. \end{cases} \tag{3}$$

Now, we give an auxiliary lemma which allow us to study the asymptotic behavior of the normalized error $\frac{\|g-P_\varepsilon(g)\|_{p,\varepsilon}}{\varepsilon^{q-1}}$.

Lemma 3. *Let $1 < p \le \infty$. Then*
a) For all net $\{Q_\varepsilon\} \subset \Pi^n$ uniformly bounded, it verifies

$$\frac{\|g - Q_\varepsilon\|_{p,\varepsilon}}{\varepsilon^{q-1}} = \frac{\|Q_\varepsilon^*\|_{p,\varepsilon}}{\varepsilon^{q-1}} + o(1), \quad \text{as } \varepsilon \to 0. \tag{4}$$

b) $\frac{\|g-P_\varepsilon(g)\|_{p,\varepsilon}}{\varepsilon^{q-1}} = O(1), \quad$ *as* $\varepsilon \to 0$.

Next we introduce k real functions on $\mathbb{R}^q$. If $1 < p < \infty$ and $1 \le i \le k$, we define

$$F_{p,i}(c_{0,i}, c_{1,i}, \ldots, c_{q-1,i}) = \int_{[-1,0]} \left| u_i t^{q-1} - \sum_{j=0}^{q-1} c_{j,i} t^j \right|^p \frac{dt}{2k} + \int_{[0,1]} \left| v_i t^{q-1} - \sum_{j=0}^{q-1} c_{j,i} t^j \right|^p \frac{dt}{2k}, \tag{5}$$

and for $p = \infty$ we define

$$F_{p,i}(c_{0,i}, c_{1,i}, \ldots, c_{q-1,i}) = \max\left\{ \max_{t \in [-1,0]} \left| u_i t^{q-1} - \sum_{j=0}^{q-1} c_{j,i} t^j \right|, \max_{t \in [0,1]} \left| v_i t^{q-1} - \sum_{j=0}^{q-1} c_{j,i} t^j \right| \right\}. \tag{6}$$

The function $F_{p,i}$ achieves a minimum on $\mathbb{R}^q$, and we write

$$B_{p,i} = \min_{\mathbb{R}^q} F_{p,i}(c_{0,i}, c_{1,i}, \ldots, c_{q-1,i}), \quad 1 \le i \le k. \tag{7}$$

If B_p is the minimum of $\sum_{i=1}^{k} F_{p,i}(c_{0,i}, c_{1,i}, \ldots, c_{q-1,i})$, where the minimum is taken over all the k-tuples of vectors in $\mathbb{R}^q$, clearly $B_p = \sum_{i=1}^{k} B_{p,i}$.

The following lemma gives us an expression for the limit of the normalized error. It is a consequence of Lemma 3 and the definition of B_p.

Lemma 4. *Let $P_\varepsilon(g)$ be a best $\|\cdot\|_{p,\varepsilon}$-approximant to g from Π^n. Then*

$$\lim_{\varepsilon\to 0} \frac{\|g - P_\varepsilon(g)\|_{p,\varepsilon}}{\varepsilon^{q-1}} = B_p^{1/p}, \quad \text{if } 1 < p < \infty,$$

and

$$\lim_{\varepsilon\to 0} \frac{\|g - P_\varepsilon(g)\|_{p,\varepsilon}}{\varepsilon^{q-1}} = B_p, \quad \text{if } p = \infty.$$

Lemma 5. *Let $1 < p \le \infty$. Then there exists a unique q-tuple $(c_{0,i}(p), \ldots, c_{q-1,i}(p))$ that minimizes (7). Moreover, $c_{q-1,i}(p) = \frac{f_-^{(q-1)}(x_i) + f_+^{(q-1)}(x_i) - 2H_0^{(q-1)}(x_i)}{2}$.*

Finally, using Lemmas 1–5 we get Theorem 3.

In [12] it was also shown that for a certain class of functions, the condition "f is differentiable up to order $q-2$" is necessary for the existence of a L^2 best local approximant. More precisely,

Theorem 4. *Let $p = 2$ and $k = 1$. Let f be a function with lateral derivatives in the ordinary sense up to order n at x_1 and suppose that there exists the best local approximant of f from Π^n on the set $\{x_1\}$. Then f is differentiable in the ordinary sense up to order $n-1$ at x_1.*

To prove Theorem 4, without loss of generality we suppose $x_1 = 0$, and we use the characterization of the best approximant from Π^n in L^2,

$$\int_{[-\varepsilon,\varepsilon]} (f - P_\varepsilon(f))(x) x^i dx = 0, \quad 0 \le i \le n.$$

Then the last integral is divided into two integrals to left and right of zero. In each one, the function f is expanded by its corresponding Taylor polynomial at zero of degree n.

As a consequence we get a linear system where the unknowns are functions of the left and right derivatives. Furthermore, the principal matrices of the system are monomials in the variable ε, and the matrices of non-homogeneous terms are certain functions of the variable ε. Finally, by an inductive argument and using Cramer's rule it can be proved that left and right derivatives of f are equal up to order $n-1$.

3. Best local simultaneous approximation

In this Section we give some results, which are proved in [9] and [10], about best local simultaneous approximation at one point to two functions. Henceforward, in this Section $A_a = [x_1 - a, x_1 + a]$.

We begin establishing the following lemma.

Lemma 6. *Let $1 < p,q < \infty$ and $f_j \in \mathscr{C}(A_a), 1 \le j \le l$. Suppose that $P \in \Pi^n$ is a $l_q - L^p$ best simultaneous approximant to f_j, $1 \le j \le l$, from Π^n on A_a, and $P \ne f_j$ for all $1 \le j \le l$. Then for all $Q \in \Pi^n$*

$$\sum_{j=1}^{l} \beta_j \left(\int_{A_a} |f_j - P|^{p-1} \operatorname{sgn}(f_j - P) Q dx \right) = 0, \tag{8}$$

where $\beta_j = \int_{A_a} |(f_j - P)(x)|^p dx)^{\frac{q}{p}-1}$, $1 \le j \le l$.

We write

$$\alpha_j = (\beta_j)^{\frac{1}{p-1}} \left(\sum_{1 \le l \le k} \beta_l^{\frac{1}{p-1}} \right)^{-1}, \quad j = 1,2,$$

where β_j is as introduced in (8).

We recall that a polynomial $P \in \Pi^n$ interpolates to a function h in the $n+1$ points, $t_0 \le t_1 \le \cdots \le t_n$, if $t_{s-1} < t_s = \cdots = t_r < t_{r+1}$, for some integer numbers s and r, $0 \le s \le r \le n$, then

$$P^{(j)}(t_s) = h^{(j)}(t_s), \quad 0 \le j \le r - s.$$

Here we put $t_{-1} = -\infty$ and $t_{n+1} = \infty$.

As a consequence of Lemma 6 we get the following interpolation result.

Theorem 5. *Let $1 < p,q < \infty$, $f_j \in \mathscr{C}(A_a)$, $j = 1,2$, and let $P \in \Pi^n$ be a $l_q - L^p$ best simultaneous approximant to f_j, $j = 1,2$, from Π^n. Then there exists j such that $P = f_j$ on a positive measure subset of A_a or P interpolates to $\alpha_1 f_1 + \alpha_2 f_2$, in at least $n+1$ different points of A_a. In addition, if $q = p$ then $\alpha_1 = \alpha_2 = \frac{1}{2}$.*

Remark 1. We observe that Theorem 5 can not be extended to the case where A_a is a union of two or more intervals. In fact, in [16] there is an example for $f_1 = f_2$, $k = 2$, $n = 1$ and $p = 2$, where the best approximant does not interpolate to f_1 at two points on A_a.

Next, we establish a uniqueness result of best simultaneous approximant in order to establish a corollary of Theorem 5.

Lemma 7. *Suppose that $f_j \in \mathscr{C}(A_a)$, $j = 1,2$, $1 < p < \infty$, and let V be a finite dimension subspace of $\mathscr{C}(A_a)$. Then there is a unique $l_\infty - L^p$ best simultaneous approximant to f_j, $j = 1,2$, from V.*

Some considerations about interpolation polynomials are also necessary. We recall Newton's divided difference formula for the interpolation polynomial of a function $h(x)$ of degree n at $y_0 \le y_1 \le \cdots \le y_n$ (see [17, 7]),

$$P(x) = h(y_0) + (x - y_0)h[y_0, y_1] + \cdots + (x - y_0)\ldots(x - y_{n-1})h[y_0, \ldots, y_n].$$

Here $h[y_0, \ldots, y_m]$ denotes the m-th order Newton divided difference. If h has continuous derivatives up to order m in $[a,b]$ containing to $y_0, \ldots, y_m$, then the m-th divided difference can be expressed as

$$h[y_0, \ldots, y_m] = \frac{h^{(m)}(\xi)}{m!},$$

for some ξ in the interval $[y_0, y_m]$. It is well known that the m-th divided difference is a continuous function as function of their arguments $y_0, \ldots, y_m$.

Next, we establish a general theorem about convergence of best approximants, which is known as Pólya algorithm. A proof is given in [15].

Theorem 6. *Let X be a real linear space, and V an n-dimensional subspace. Suppose $\|\cdot\|_k$ ($1 \le k < \infty$), $\|\cdot\|$, are norms on $V \bigoplus \{y\}$, where $y \in X \backslash V$, and $\|x\|_k$ converges to $\|x\|$ for all $x \in X$. Let p_k be a best approximant to y from V with respect to $\|\cdot\|_k$, p a best approximant with respect to $\|\cdot\|$. Then the set of cluster points of $\{p_k : 1 \le k < \infty\}$ is nonempty and contained in the set of best approximants with respect to $\|\cdot\|$; furthermore, if p is unique, $\|p_k - p\|$ converges to zero.*

To prove the following corollary of Theorem 5 we use the continuity of the divided difference, Lemma 7 and Theorem 6.

Corollary 1. *Let $f_1, f_2 \in \mathscr{C}^n(A_a)$. If $1 < p \le \infty$, and $P \in \Pi^n$ is the $l_\infty - L^p$ best simultaneous approximant to f_j, $j = 1,2$, from Π^n, then there are points in A_a, $t_0 \le t_1 \le \cdots \le t_n$, such that P interpolates to some convex combination of f_j, $j = 1,2$, at the points t_i, $0 \le i \le n$.*

The following lemma is of general character and it was proved in [1].

Lemma 8. *Let $(X, \|\cdot\|)$ be a normed linear space. Let S be a finite dimension linear subspace of X and let $f_1, f_2 \in X$. If $P \in S$ is a $l_\infty - \|\cdot\|$ best simultaneous approximant to f_1 and f_2, i.e., P minimizes*

$$E(Q) := \max\{\|f_1 - Q\|, \|f_2 - Q\|\}, \quad Q \in S,$$

then $E(P) = \|f_1 - P\| = \|f_2 - P\|$ or P is a best approximant to the function where $E(P)$ is attained.

In [10] it was proved an estimate of the error

$$E_\varepsilon := \max\{\|f_1 - P_\varepsilon\|_{p,\varepsilon}, \|f_2 - P_\varepsilon\|_{p,\varepsilon}\}.$$

Theorem 7. *Let $f_1, f_2 \in \mathscr{C}^{n+1}(A_a)$. It verifies*

$$E_\varepsilon = \frac{1}{2}\|T(f_1) - T(f_2)\|_{p,\varepsilon} + O(\varepsilon^{n+1}) \quad \text{as } \varepsilon \to 0,$$

where $T(f_j)$, $j = 1,2$, is the Taylor polynomial of f_j of degree n at x_1.

As a consequence of Lemma 8 and Theorem 7, it was proved in [10] the following partial result about the convergence of the best simultaneous approximants.

Theorem 8. *Let $1 < p < \infty$, $f_j \in \mathscr{C}^{n+1}(A_a)$, $j = 1,2$, and let $P_\varepsilon(f_1, f_2)$ be the $l_\infty - L^p$ best simultaneous approximant to f_j from Π^n on A_ε. If*

$$\|f_1 - P_\varepsilon(f_1, f_2)\|_{p,\varepsilon} \neq \|f_2 - P_\varepsilon(f_1, f_2)\|_{p,\varepsilon}$$

for some net $\varepsilon \downarrow 0$, then $T(f_1) = T(f_2)$ and $P_\varepsilon(f_1, f_2) \to T(f_1)$ as $\varepsilon \to 0$.

Also, in [10] information is given about the behavior of the coefficients of the polynomials $P_\varepsilon(f_1, f_2)$ when $\|f_1 - P_\varepsilon(f_1, f_2)\|_{p,\varepsilon} = \|f_2 - P_\varepsilon(f_1, f_2)\|_\varepsilon$.

Theorem 9. *Let $f_j \in \mathscr{C}^{n+1}(A_a)$, $j = 1,2$, $T(f_1, f_2) := \frac{T(f_1)+T(f_2)}{2}$, and $-1 \le m \le n-1$. Let $P_\varepsilon(f_1, f_2)$ be the $l_\infty - L^p$ best simultaneous approximant to f_j from Π^n on A_ε. Suppose that*

$$\|f_1 - P_\varepsilon(f_1, f_2)\|_{p,\varepsilon} = \|f_2 - P_\varepsilon(f_1, f_2)\|_{p,\varepsilon}, \quad 0 < \varepsilon \le \varepsilon_0.$$

If $f_1^{(i)}(x_1) = f_2^{(i)}(x_1)$, $0 \le i \le m$, we have

$$\lim_{\varepsilon \to 0} P_\varepsilon^{(i)}(x_1) = T(f_1, f_2)^{(i)}(x_1), \quad 0 \le i \le m+1$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{i-m-1} P_\varepsilon^{(i)}(x_1) = 0, \quad m+1 < i \le n.$$

Using the interpolation properties of the best simultaneous approximants, it is studied in [9] the convergence of the best simultaneous approximants when the measure of the interval tends to zero.

Theorem 10. *Let $1 < p < \infty$. Suppose that $f_j \in \mathscr{C}^n(A_a), j = 1,2$, and let $P_\varepsilon(f_1, f_2)$ be the $l_p - L^p$ best simultaneous approximant to f_j, from Π^n on A_ε. Then*

$$P_\varepsilon(f_1, f_2) \to T(f_1, f_2), \quad as\ \varepsilon \to 0,$$

where $T(f_j)$, $j = 1,2$, is the Taylor polynomial of f_j of degree n at x_1.

Under other conditions of differentiability we obtain a similar result for the infinite and l_q norms in $\mathbb{R}^2$.

Theorem 11. *Let $1 < p, q \leq \infty$ and let $f_j \in \mathscr{C}^{n+1}(A_a), j = 1,2$. Let $P_\varepsilon(f_1, f_2)$ be the $l_\infty - L^p$ best simultaneous approximant to f_j, $j = 1,2$, from Π^n on A_ε. Then*

$$P_\varepsilon(f_1, f_2) \to T(f_1, f_2), \quad as\ \varepsilon \to 0.$$

4. Best local simultaneous approximation in L^2

In this Section we give a result of best local simultaneous approximation in several points when we consider $l_\infty - L^2$ best approximation. So, in this Section $k \geq 1$.

Here, it is not necessary the property of interpolation, which as we have mentioned in Remark 1 is not valid in general if $k > 1$. In fact, we use the following result that was proved in [14] for spaces with an inner product.

Theorem 12. *Let X be a real inner product space and G a subspace of X. Consider two elements $y_1, y_2 \in X$, which have best approximants, say g_1, g_2, from the elements of G. Then there exists $\lambda \in [0,1]$ such that*

$$g_0 = \lambda g_1 + (1 - \lambda) g_2,$$

is the best simultaneous approximant of y_1 and y_2.

The next theorem is a consequence of Theorem 2 and Theorem 12.

Theorem 13. *Let $n + 1 = kq$, $f_j \in \mathscr{C}^{n+1}(A_a)$, $j = 1,2$, and let $P_\varepsilon(f_1, f_2)$ be the $l_\infty - L^2$ best simultaneous approximant to f_j, $j = 1,2$, from Π^n on A_ε. Then*

$$P_\varepsilon(f_1, f_2) \to H(f,g) := \frac{H(f_1) + H(f_2)}{2} \quad as\ \varepsilon \to 0,$$

where $H(f_j) \in \Pi^n$, $j = 1,2$ is determined by the conditions

$$H(f_j)^{(l)}(x_i) = f_j^{(l)}(x_i), \quad 1 \leq i \leq k,\ 0 \leq l \leq q - 1.$$

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